



Topics in Algorithms 2005

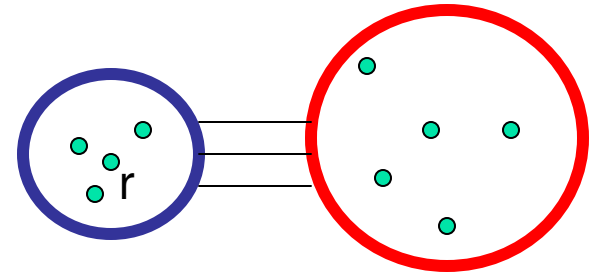
Disjoint Paths, Cuts, Arborescences

Ramesh Hariharan



Undirected Unweighted Cuts

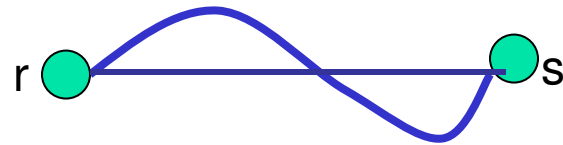
- **Cut:** Partition of the vertex set
- **Cut Size:** Number of edges between the two sides of the partition





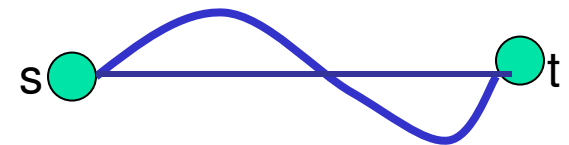
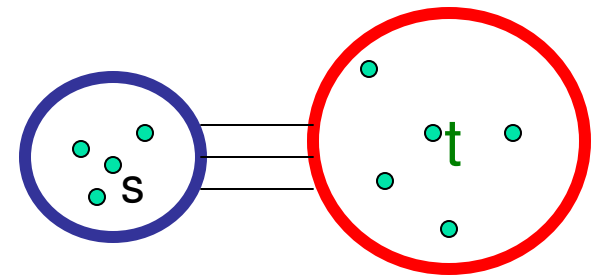
Undirected Unweighted Cuts

- **Edge Connectivity between s and t :** The number of edge-disjoint paths between s and t



Menger's Theorem

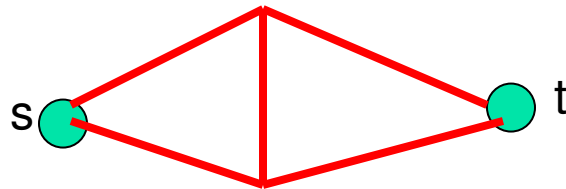
- The size of the smallest cut separating r and s equals the number of edge disjoint paths between s and t





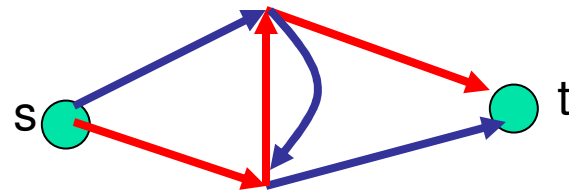
Proving Menger's Theorem

- Find paths from s to t repeatedly
- Stop when no more paths can be found
- Does this work?



Proving Menger's Theorem

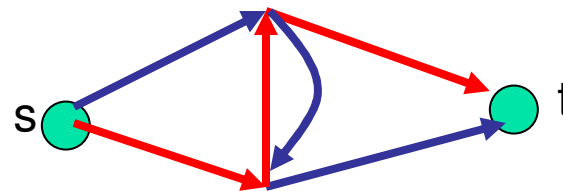
- Need to allow undoing of wrong decisions
- The Trick: Replace undir edge by 2 dir edges
 - An edge used in one direction is marked; it can still be used but only in the other direction
 - An edge used in both directions is unmarked; it behaves as if it was never used





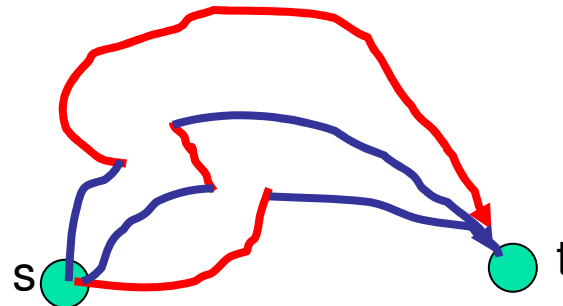
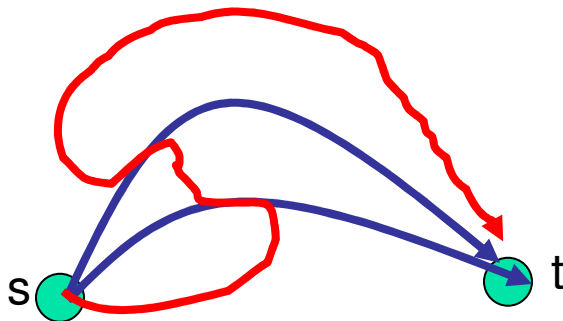
Proving Menger's Theorem

- Find directed paths repeatedly
- A path found must respect marking, i.e., it uses marked edges only in the unique available direction; unmarked edges can go in both directions
- From these directed paths, how do we get our undirected paths?
 - Problem: each undirected edge needs to be used exactly one over all paths; however, several directed paths may use a particular undirected edge



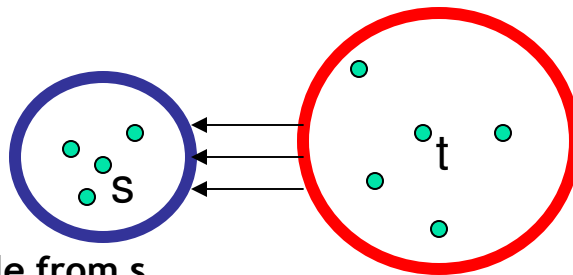
Proving Menger's Theorem

- By induction
- Suppose one has k paths
- Assume you can convert $k-1$ from directed to “undirected” maintaining the following
 - Edges used an even number of times amongst these $k-1$ paths are not used at all by these paths. Edges used an odd number of times are used exactly once and only in the “right” direction
- Now take the k th directed path and throw it in as shown below: REMEMBER to maintain the odd/even criterion above



Proving Menger's Theorem

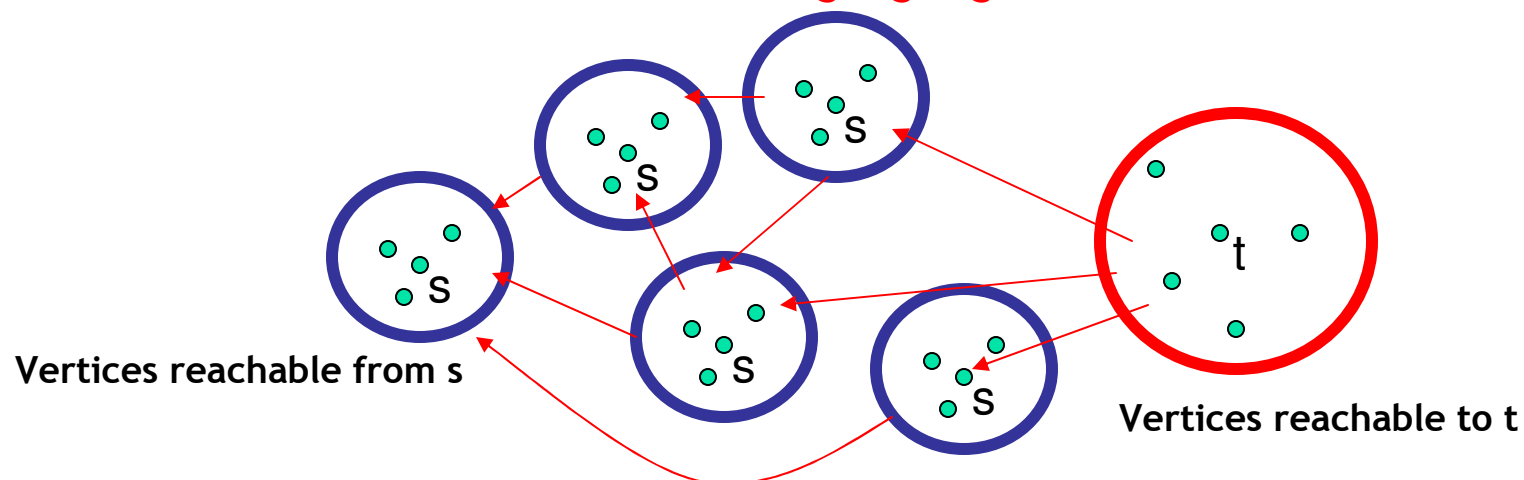
- Fine, so we have as many undirected paths as directed paths
- But why do we have sufficiently many directed paths (as many as the min-cut)?
- Suppose not: so you run into a deadend, can't find any more paths
- Now demonstrate a cut of size as much as the number of paths found; how?
 - Take all vertices reachable from s in the unmarked graph
 - Claim: This is a cut of size equal to the number of paths found so far
 - Why?? **Exercise**



Vertices reachable from s

Capturing All s-t Min-Cuts

- The residual graph (comprising unused directed edges; if unmarked both directions are unused, otherwise only one direction is unused) has components
- Vertices reachable from s
- Vertices reachable to t
- Strongly connected components comprising vertices other than s or t
- The residual graph can be thought of as a DAG on these components
- Any s-t cut in this DAG with all edges going towards s is a mincut: Exercise??
- All s-t cuts are mincuts in this DAG with edges going towards s: Exercise??



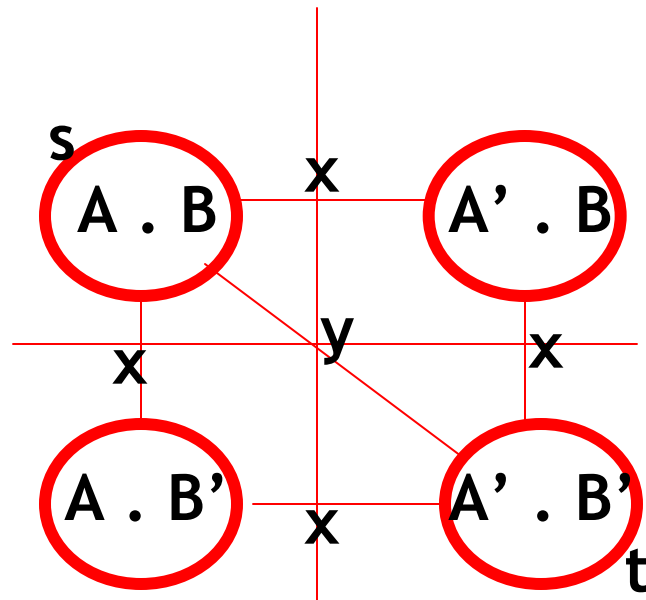


How many s-t min-cuts can be there?

- Take n edges organized into k disjoint paths of length a, b, c, \dots adding up to n
- How many cuts does this give??
- Maximize over a, b, c adding up to n

s-t cuts Nesting

- If you have two s-t min cut, show that their intersection is also a s-t mincut: **Exercise??**



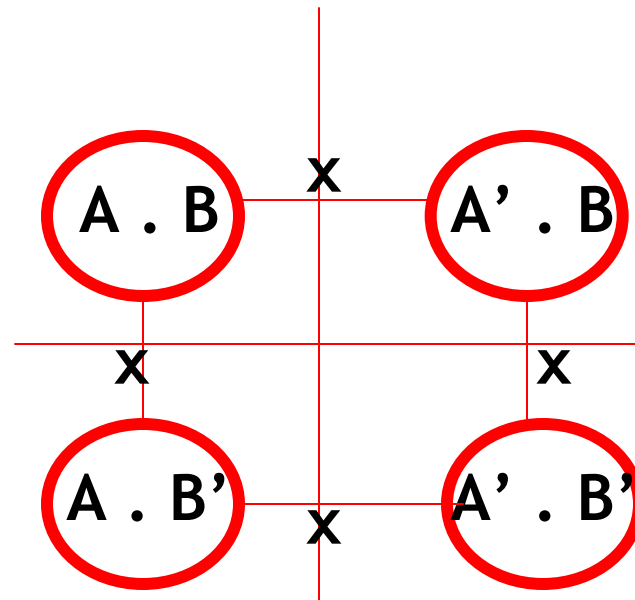


Global Min-Cuts

- Take s - t min-cuts over all s, t pairs;
- Cuts of the smallest size are called global min-cuts
- How many global min-cuts are there?

Global Min-Cuts Nesting

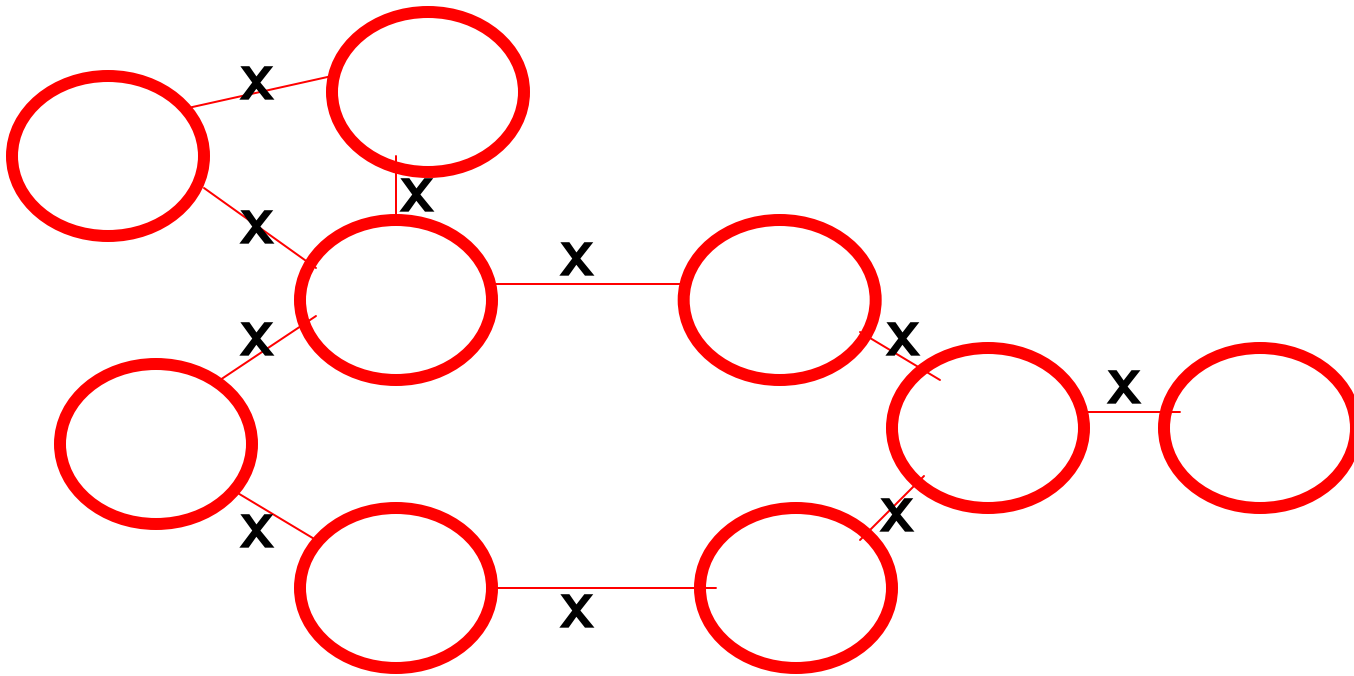
- If you have two global min cuts, show that all 4 cuts below are global min-cuts. **Exercise??**





The Cactus of Global Cuts

- All global min-cuts form a cactus. Any disconnection of the cactus is a min-cut. **Exercise?? How many global min-cuts does this imply.**





Finding a global min-cut

- A simple randomized algorithm by Karger
 - Take a random edge and contract it
 - What is the probability that you are left with a min-cut at the end
 - So no edge of the min-cut should get contracted
 - Prob of this hapenning in the first round is $1 - k/e > (n-2)/n$. Why??
 - Overall probability of success = $1 / \binom{n}{2}$
 - Repeat n^2 times for success
 - Time taken: $O(n^2 * \log n)$
- Implementing one phase
 - Take a random permutation of the edges
 - Run what??



Edmonds Arborescences

- Analogues of Menger's Theorem for Global Connectivity
- Take an arbitrary s as root
- If global min-cut is k , then there are k edge disjoint arborescences.

- Arborescence: Spanning tree with all edges directed away from the root; as before we need to direct the graph by replacing edges in both directions.

- Example: take a simple cycle



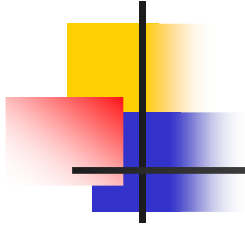
Edmonds Arborescences: Proof

- Existence Proof: Next Class
- Open Problem: Construct these arborescences in sub quadratic time, even for the case $k=2$



References

- Papers by David Karger on Min-Cuts, available on the web
- Edmonds Theorem, Cactus Structures: Still trying to find good references, will provide next time



Thank You
