

# Short-cuts on Star, Source and Planar Unfoldings

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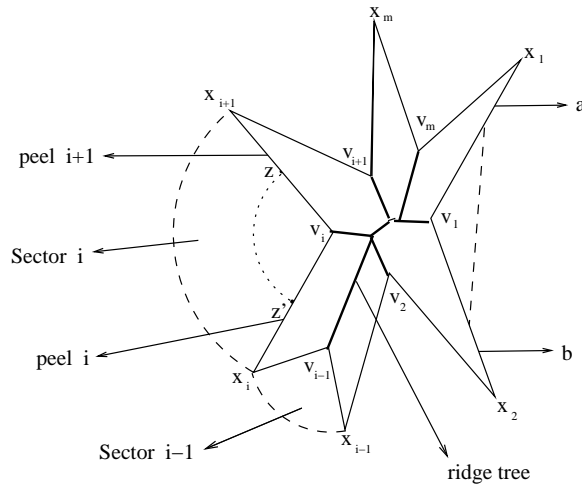
**Abstract.** When studying a 3D polyhedron, it is often easier to cut it open and flatten in on the plane. There are several ways to perform this unfolding. Standard unfoldings which have been used in literature include *Star* Unfoldings, *Source* Unfoldings, and *Planar* Unfoldings, each differing only in the cuts that are made. Note that every unfolding has the property that a straight line between two points on this unfolding need not be contained completely within the body of this unfolding. This could potentially lead to situations where the above straight line is shorter than the shortest path between the corresponding end points on the polyhedron. We call such straight lines *short-cuts*. The presence of short-cuts is an obstacle to the use of unfoldings for designing algorithms which compute shortest paths on polyhedra. We study various properties of Star, Source and Planar Unfoldings which could play a role in circumventing this obstacle and facilitating the use of these unfoldings for shortest path algorithms.

We begin by showing that Star and Source Unfoldings do not have short-cuts. We also describe a new structure called the *Extended Source* Unfolding which exhibits a similar property. In contrast, it is known that Planar unfoldings can indeed have short-cuts. Using our results on Star, Source and Extended Source Unfoldings above and using an additional structure called the *Compacted Source* Unfolding, we provide a necessary condition for a pair of points on a Planar Unfolding to form a short-cut. We believe that this condition could be useful in enumerating all Shortest Path Edge Sequences on a polyhedron in an output-sensitive way, using the Planar Unfolding.

## 1 Introduction

*Star Unfoldings* have proved useful tools in arguing about shortest paths on the surface of a 3D polytope  $\mathcal{P}$  (see Fig. 1(a)). A Star Unfolding is obtained by cutting a 3D polytope along shortest paths emanating from a reference point  $x$  and going to each of the vertices of  $\mathcal{P}$ ; the resulting structure is then flattened on the plane. We denote the Star Unfolding with respect to the reference point  $x$  by  $star(x)$ . The idea of such an unfolding is due to Aleksandrov [1], who uses it to show that  $\mathcal{P}$  can be triangulated. Aronov and O'Rourke [10] showed that this unfolding does not self-overlap (in fact they show a stronger property, which we shall describe shortly). Subsequently, Agarwal, Aronov, O'Rourke and Schevon

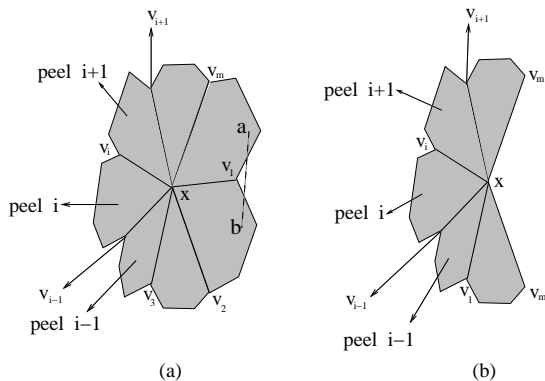
[12] showed how the Star Unfolding can be used for enumerating Shortest Path Edge Sequences on a convex polyhedron and for finding the geodesic diameter of a convex polyhedron. Chen and Han [11], independently, used Star Unfoldings for computing shortest path information from a single fixed point on the surface of a polytope.



**Fig. 1.** Star Unfolding

The *Source Unfolding* of a 3D polytope  $\mathcal{P}$  is obtained by transforming the Star Unfolding as follows. The Star Unfolding  $star(x)$  (with respect to the reference point  $x$ ) has a *ridge tree* [7] comprising all points which have two or more distinct shortest paths from the reference point  $x$ . The leaves of this ridge tree are the vertices of  $\mathcal{P}$  ( $x$  aside) and internal ridge nodes are those points on  $\mathcal{P}$  which have 3 or more shortest paths to  $x$ . Cutting along the ridge tree breaks the Star Unfolding into several convex polygons, called *peels*. The Source Unfolding is obtained by rearranging these peels around  $x$ . We denote the Source Unfolding with respect to the reference point  $x$  by  $source(x)$ . In Fig. 2(a) and (b), we have shown the Source Unfolding of  $\mathcal{P}$  with respect to  $x$  for the cases when  $x$  is not a vertex on  $\mathcal{P}$  and when  $x$  is a vertex of  $\mathcal{P}$ , respectively. This unfolding was defined in [3]. Note that in case 1,  $x$  is completely surrounded by the body of the unfolding while in the other case, there is an empty angular region incident on  $x$  due to the positive curvature of the polyhedron at  $x$ .

We define *Planar Unfoldings* as follows. Consider any sequence of distinct faces  $f_1 \dots f_k$  on the surface of  $\mathcal{P}$  such that  $f_i$  and  $f_{i+1}$  share an edge  $e_i$ , for all  $i$ ,  $1 \leq i \leq k - 1$ . Consider cutting out  $f_1 \dots f_k$  from  $\mathcal{P}$  by cutting along each edge that is on the boundary of some  $f_i$ , with the exception of the edges  $e_1 \dots e_{k-1}$ . Note that by virtue of  $f_1 \dots f_k$  being distinct, cuts are made along



**Fig. 2.** Source Unfolding

all but exactly two of the boundary edges for each  $f_i$ ,  $2 \leq i \leq k-1$ , (the exceptions being  $e_{i-1}, e_i$  for  $f_i$ ). Further, for  $f_1$  and  $f_k$ , cuts are made along all but exactly one of the boundary edges (the exceptions being  $e_2, e_{k-1}$ ). With these properties, it is easy to see that the resulting structure can be flattened out on a plane (because, after cutting out,  $f_i$  and  $f_{i+1}$  are attached to each other by exactly one edge, around which we can rotate to get  $f_i$  and  $f_{i+1}$  on one plane). We call this structure the Planar Unfolding with respect to the edge sequence  $e_1 \dots e_{k-1}$ .

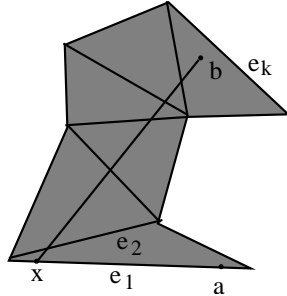
**Our Results.** Our contribution is four-fold, all using only elementary geometry.

**1.** Aronov and O'Rourke [10] showed that the Star Unfolding does not self-overlap. Further, they showed a stronger property: the circular sectors between adjacent peels (see Fig.1(a)) are also disjoint. As a corollary of this property, they show that there are no short-cuts in the Star Unfolding with one endpoint being the reference point  $x$ . In other words, a straight line between any instance of  $x$  in the Star Unfolding and any other point  $a$  in the Star Unfolding is no shorter than the shortest path between  $x$  and  $a$  on  $\mathcal{P}$ . We extend this result to all pairs of points on the Star Unfolding. So, for any pair of points  $a$  and  $b$  on the Star Unfolding, we show that the straight line joining these two points is no shorter than the shortest path between  $a$  and  $b$  on  $\mathcal{P}$ .

**2.** Using the above result that the Star Unfolding has no short-cuts, we show that the Source Unfolding also does not have short-cuts. The proof uses a stepwise transformation of the Star Unfolding into the Source Unfolding.

**3.** Next, using the above result for Source Unfoldings, we show that the extension (with respect to any point  $y$ ) of the Source Unfolding does not have short-cuts, i.e., for any point  $a$  on the extended line  $xy$  in the Extended Source Unfolding and any other point  $b$  on the Source Unfolding, the straight line joining  $a$  and

$b$  is no shorter than the shortest path between  $a$  and  $b$  on  $\mathcal{P}$ . This notion of *Extended Source Unfoldings* will be defined later.



**Fig. 3.** Planar Unfolding

4. It is known [5] (and we give an example to this effect) that, unlike Star and Source Unfoldings, Planar Unfoldings could have short-cuts, i.e., the distance between a pair of points  $x$  and  $y$  on a Planar Unfolding can be shorter than the distance between the same pair of points on the convex polyhedron. However, the above result on Extended Source Unfoldings gives the following necessary condition for a pair of points to form a short-cut on a Planar Unfolding.

**A Necessary Condition for a Short-Cut on Planar Unfoldings.** Consider three points  $x, a, b$  on some Planar Unfolding of  $\mathcal{P}$  (see Fig.3) such that  $xa$  and  $xb$  are images of geodesics completely within this unfolding. For  $ab$  to be a short-cut in this Planar Unfolding, neither  $xa$  nor  $xb$  must be a shortest path on  $\mathcal{P}$ .

In the above process, we also define a new structure called the *Compacted Source Unfolding* by refolding the Source Unfolding in a certain way, and then use this structure to show properties of shortest paths which are then used to prove the above claims on Star, Source, Extended Source and Planar Unfoldings.

**Applications and Further Work.** Our motivation for studying short-cuts was to obtain an output-sensitive algorithm for enumerating SPES's (i.e., shortest path edge sequences, sequences of edges which are crossed by at least one shortest path) on the polyhedron  $P$ . Unlike previous work [12, 7] which uses the Star Unfolding as its main tool, our approach was to use the Planar Unfolding, where a particular edge sequence is considered if and only if all its prefixes are SPES's; this would lead naturally to output-sensitivity, as only SPES's are considered for further extension using a Dijkstra like scheme. The first hurdle encountered in this process is that of short-Cuts. While we haven't yet obtained a complete output-sensitive algorithm for enumerating SPES's, we believe that the properties of short-cuts listed above could be useful in obtaining such an algorithm. In

addition, in a companion paper [13, 14], we make heavy use of these properties to prove some more interesting properties of Planar Unfoldings, described below.

Consider a Planar Unfolding of the faces  $f_1 \dots f_k$  forming the edge sequence  $e_0 \dots e_k$ , where  $e_i$  is common to  $f_i$  and  $f_{i+1}$ , for all  $i$ ,  $1 \leq i \leq k-1$ . Further, suppose  $e_0 \dots e_{k-1}$  is an SPES crossed by shortest path  $a, b$  and  $e_1 \dots e_k$  is an SPES crossed by shortest path  $a', b'$ . Then ALL short-cuts between  $e_0$  and  $e_k$  lie completely on one particular side of the infinite straight line passing through  $a$  and  $b$  (wlog, keeping this line vertical with  $a$  below  $b$ , short-cuts are either all to the left of this line or they are all to the right of this line). Further, if  $xy$  is a short-cut to the right of  $ab$  with  $x \in e_0$  and  $y \in e_k$ , then either  $source(a)$  has all shortest paths  $ay$  in the clockwise region between the shortest paths  $ab'$  and  $ax$ , or  $source(b')$  has all shortest paths  $b'x$  in the clockwise region between the shortest paths  $b'y$  and  $b'a'$ ,

**Road map.** Section 2 lays out some preliminaries. Section 3 defines Compacted Source Unfoldings and proves some properties needed in later sections. Sections 4, 5 and 6, prove that there are no short-cuts on Star, Source and Extended Source Unfoldings, respectively. Section 11 provides an example for the presence of short-cuts in Planar Unfoldings. Section 7 shows our necessary condition for a pair of points on a Planar Unfolding to form a short-cut. For want of space, Section 11,

## 2 Definitions and Preliminaries

Let  $\mathcal{P}$  denote a 3D polyhedron. We use the term *vertex* of  $\mathcal{P}$  to denote one of the corners of  $\mathcal{P}$ . For any two points  $a, b$  on  $\mathcal{P}$ , let  $SP(a, b)$  denote the shortest path between  $a$  and  $b$  on  $\mathcal{P}$ . For any two points  $a, b$  on the plane, let  $ab$  denote both the straight line segment connecting  $a, b$  and the length of this straight line segment as well (for convenience). Let  $C(a, ab)$  denote the circle with center  $a$  and radius  $ab$ .

**Definitions.** For any two points  $a, b$  on the Star Unfolding, let  $star(a, b)$  be the length of the straight line  $ab$ .  $source(a, b)$  and  $planar(a, b)$  are defined analogously.

Aronov and O'Rourke [10] showed that the Star Unfolding can be embedded in the plane without overlap. The resulting structure is a simple polygon (see Fig.1(a)) whose vertices in cyclic order are denoted by  $x_1, v_1, x_2, v_2, \dots, x_m, v_m$ , where  $x_1 \dots x_m$  are images of the source point  $x$  and  $v_1 \dots v_m$  are the vertices of  $\mathcal{P}$  (excluding  $x$ , if it is a vertex). Note that for each  $i$ ,  $1 \leq i \leq m$ ,  $x_i v_i = v_i x_{i+1}$  (here, and in what follows,  $i+1$  is taken modulo  $m$ , implicitly). Aronov and O'Rourke actually showed the following stronger fact in Lemma 1, for which we need the following definition. Let  $Sector(i)$  denote the sector of the circle  $C(v_i, x_i v_i)$  which lies between the lines  $x_i v_i$  and  $x_{i+1} v_i$  (see Fig.1).

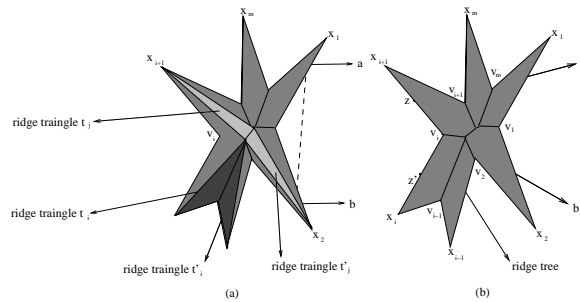
**Lemma 1.** [10] *The various sectors  $Sector(1) \dots Sector(m)$  are mutually pairwise disjoint; further these sectors lie completely outside the Star Unfolding. It follows that the Star Unfolding does not self-overlap and is a simple polygon.*

Lemma 1 yields the following lemma on peels (recall, peels were defined in the introduction).

**Lemma 2.** *For any point  $y$  on  $\text{star}(x)$ , the Star Unfolding with respect to  $x$ , there exists an  $x_i$ ,  $1 \leq i \leq m$ , such that the straight line  $x_i y$  is contained completely within the peel containing  $x_i$ ; further,  $\text{star}(x_i y) = SP(xy)$ .*

**Fact 1** *Consider two points  $a, b$  on either a Star, Source or Planar Unfolding such that the straight line joining them lies completely inside the unfolding (see Fig.1(b) for an example). Then, this straight line is an image of a geodesic on the surface of  $\mathcal{P}$  and therefore, has length at least  $SP(a, b)$ .*

**Triangulating  $\text{star}(x)$  and Triangle Images.** Note that if we triangulate the peels in  $\text{star}(x)$  using triangles whose one endpoint is always some  $x_i$ , then each such triangle has a congruent *image triangle*; this is illustrated in the picture below where  $t'_j$  is the image of  $t_j$  and  $t'_i$  is the image of  $t_i$ . This notion of image of a triangle will be used in defining Compacted Source Unfoldings in the next section. Note that the image pairing of triangles has a nesting property, namely that the various angular stretches between triangles and their respective images are either mutually disjoint or one stretch contains the other.

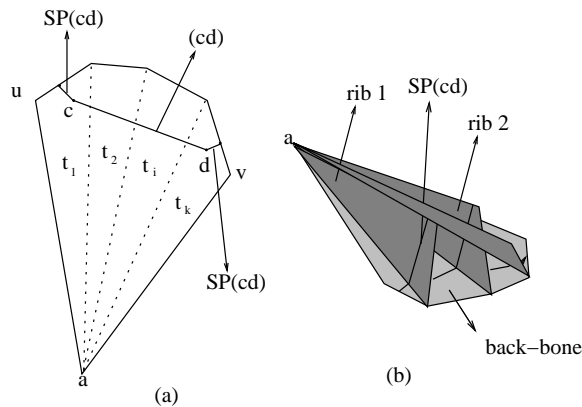


**Fig. 4.** Star Unfolding: Triangle Images

### 3 The Compacted Source Unfolding

In this section we define a new structure called the *Compacted Source Unfolding* which is obtained by folding the Source Unfolding as follows. Consider  $\text{source}(a)$ , the Source Unfolding with respect to  $a$  and suppose one is given any two points  $u$  and  $v$  on this unfolding such that  $SP(au)$  and  $SP(av)$  are maximal shortest paths in  $\text{source}(a)$ . Consider the region of  $\text{source}(a_{uv})$  which lies clockwise from  $SP(au)$  to  $SP(av)$  and consider the triangles into which peels in this region are

triangulated. Some of these triangles will have their images within this region as well, while others will not. Fold  $source(a_{uv})$  in such a way that each triangle which has its image in this region gets pasted to its image, back to back (this is possible because of the nesting property described above). The resulting structure will have a *backbone*, comprising those triangles which do not have images in this region, and *ribs* comprising triangles which indeed have images in this region, with the images pasted back to back; this is the *Compacted Source unfolding* with respect to  $a$  and  $u, v$ .



**Fig. 5.** The Compacted Source Unfolding: Backbones and Ribs

We can show the following properties on this Compacted Source Unfolding. Theorem 1 and its corollary below follow from the fact that a path which goes through the ribs can be shortened to one that stays on the backbone but has the same source and destination. Theorem 2 is the key non-trivial claim that we prove. Proofs appear in the Appendix.

**Lemma 3.** *The back-bone of  $source(a_{(uv)})$  is a convex polygon.*

**Theorem 1.** *Every pair of points on a peel is joined by a unique shortest path on the convex polyhedron that lies entirely on that peel.*

**Corollary 1.** *The intersection of a shortest path with a peel is connected.*

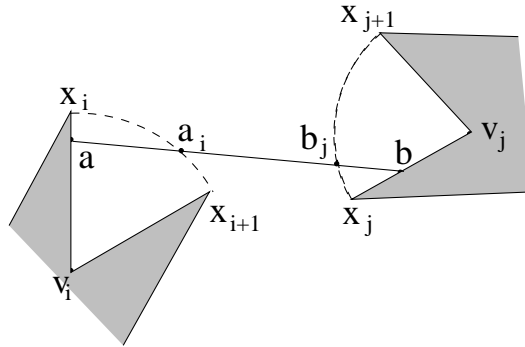
**Theorem 2.** *Consider a portion  $source(a_{(uv)})$  of  $source(a)$ , where  $u$  and  $v$  are any pair of ridge points such that the angle  $\angle vau < \pi$  (Fig.14(a)). Let  $(uv)$  denote the line segment that joins  $u$  and  $v$  on the plane that contains  $source(a)$  and suppose that  $SP(av) \geq SP(au)$ . If  $(uv)$  lies completely outside the Source Unfolding (except for its endpoints), then there exists another shortest path  $SP(au)$  in  $source(a_{(uv)})$ .*

## 4 Star Unfoldings

We prove the following theorem in this section.

**Theorem 3.** *Consider any two points  $a, b$  on the Star Unfolding with respect to point  $x$  on  $\mathcal{P}$ . Then  $star(ab) \geq SP(ab)$ .*

The proof outline is as follows: By Fact 1, this is true if the line segment  $ab$  is completely inside  $star(x)$ . It now suffices to consider the case when  $ab$  is completely outside  $star(x)$ , for the only remaining case can be handled by splitting  $ab$  into internal and external segments and arguing on each segment. For the case when  $ab$  is completely outside as shown in Fig.6, using Lemma 1, one can show that the path  $a$  to  $x_i$  and then  $x_j$  to  $b$  on the polyhedron  $\mathcal{P}$  is shorter than  $ab$ , as required. The full proof appears in the appendix.



**Fig. 6.**  $a, b$  completely outside

## 5 Source Unfoldings

**Theorem 4.** *Consider any two points  $a, b$  on the Source Unfolding with respect to point  $x$  on  $\mathcal{P}$ . Then  $source(ab) \geq SP(ab)$ .*

Note that as in the case of  $star(x)$ , we need only consider the case when the line segment  $ab$  is completely outside  $source(x)$ . The proof proceeds by induction on the number of peels (call this the *peel distance*) in  $source(x_{ab})$ . If this is one or two, then the argument is easily made by rotating peels from the source configuration to the star configuration while showing that distance  $ab$  only decreases in this process; this along with Theorem 3 finishes the proof. If the peel distance is more than two, then Theorem 2 gives us a tool to perform induction.

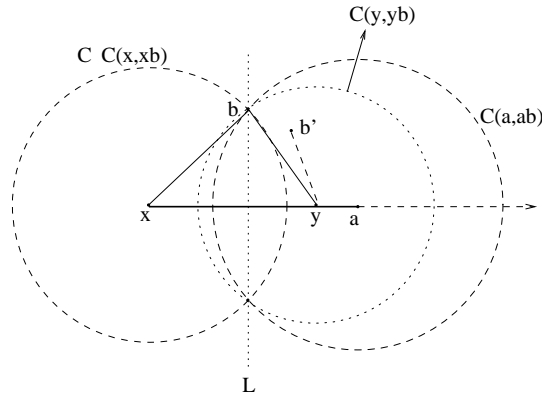
Without loss of generality, we assume that the angle at  $x$  in  $source(x_{ab})$  is  $< \pi$  and that  $xb \leq xa$  on  $source(x)$ . Then, by Theorem 2, there exists another copy  $b'$  of  $b$  in  $source(x_{ab})$ . We then argue inductively on  $ab'$  instead of  $ab$ , as the former has a provably smaller peel distance. The full proof appears in the Appendix.

## 6 Extended Source Unfoldings

We define the *Extended Source Unfolding* of  $source(x)$  constructively as the structure obtained from the Source Unfolding by taking a shortest path  $xy$  on  $source(x)$  (which also corresponds to a shortest path on the polyhedron  $\mathcal{P}$ ) and then extending it as a geodesic for a finite stretch or until it reaches a vertex of  $\mathcal{P}$ . We prove the following theorem in this section.

**Theorem 5.** *Consider the Source Unfolding  $source(x)$  with respect to  $x$ . Let  $xb$  and  $xy$  be any pair of shortest paths. Let  $a$  be any point that lies on a geodesic that is a finite extension of the shortest path  $xy$ . Then  $source(ab) \geq SP(ab)$ .*

**Proof.** Let us assume that  $xa$  in the Extended Source Unfolding is along the x-axis (in a Cartesian coordinate system). Then we have  $x, y$  and  $a$  in that order from left to right along x-axis as shown in Fig.7. We have the following two cases to consider.



**Fig. 7.**  $source(ab) \geq SP(ab)$  in extended source unfolding: Case 1

**Case 1:** The portion  $ya$  of  $xa$  is a shortest path in  $source(y)$ : Let  $source(x)$ ,  $source(y)$  and  $source(a)$  denote the Source Unfoldings of the polyhedron with respect to  $x, y$  and  $a$  respectively. Let us superimpose  $source(y)$  and  $source(a)$  on the line segment  $xa$  in the Extended Source Unfolding of  $source(x)$  such that the following conditions are satisfied.

- The portions  $xy$  (a shortest path) and  $ya$  (a shortest path) of  $xa$  in the Extended Source Unfolding coincide with the shortest paths  $xy$  and  $ya$  in  $source(y)$ . This is always possible for the following reason. Since  $xa$  is an extension of the shortest path  $xy$  in the Extended Source Unfolding,  $y$  cannot be a polyhedral vertex (according to the definition of the Extended Source Unfolding). Hence, a small neighborhood of  $y$  in the Extended Source Unfolding is identical to a small neighborhood of  $y$  in  $source(y)$ . Thus, the angle between the shortest paths  $ya$  and  $yx$  in each of the Extended Source Unfolding and  $source(y)$  is equal to  $\pi$ .
- The portion  $ya$  (a shortest path) of  $xa$  coincides with the shortest path  $ya$  in  $source(a)$ .

Let  $b'$  denote the position of  $b$  in  $source(y)$ . That is, a shortest path from  $y$  to  $b$  in  $source(y)$  is denoted by  $yb'$ . Consider the three discs  $C(x, xb)$ ,  $C(y, yb')$  and  $C(a, ab)$  (we define the disc  $C(x, xb)$  as the set of points  $p$  such that  $xp \leq xb$ , other discs are defined similarly). Since  $xy$  and  $xb$  are shortest paths, according to Theorem 4,  $xb \geq SP(yb)$ . Thus, the shortest path from  $y$  to  $b$ , that is  $yb'$ , in  $source(y)$  is shorter than  $yb$ . Thus,  $b'$  lies on the disc  $C(y, yb')$  (Fig.7). Since  $yx$  and  $yb'$  are shortest paths in  $source(y)$ ,  $xb' \geq SP(xb') = SP(xb)$ . Hence,  $b'$  does not lie inside the disc  $C(x, xb)$  (because the radius of this disc is a shortest path). Thus we have  $b'$  on the portion of the disc  $C(y, yb')$  that is not inside  $C(x, xb)$ . Hence,  $b'$  cannot lie to the left of the line segment  $L$  through  $b$  that is normal to the x-axis. But this portion of the disc  $C(y, yb')$  (that contains  $b'$ ) is on the disc  $C(a, ab)$  because  $a$  is to the right of  $y$ . Thus, we have  $b'$  on or inside  $C(a, ab)$ , that is,  $ab \geq ab'$ .

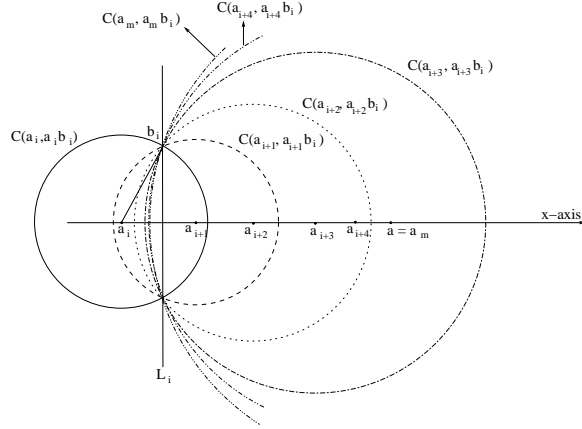
Consider the two shortest paths  $ya$  and  $yb'$  in  $source(y)$ . Since, we have assumed that the portion  $ya$  of  $xa$  is a shortest path, according to Theorem 4,  $ab' \geq SP(ab') = SP(ab)$ . Thus, we have  $ab \geq ab' \geq SP(ab)$ , and the theorem holds for this case.

**Case 2:**  $ya$  is not a shortest path in  $source(y)$ : Let us partition  $xa$  into a sequence of shortest paths (Fig.8)  $a_0a_1, a_1a_2, a_2a_3, \dots, a_{m-1}a_m$  (where  $a_0 = x$ ,  $a_1 = y$  and  $a_m = a$ ) such that, each partition  $a_i a_{i+1}$  is the portion of  $ya$  that lies on a single peel. Each of these partitions satisfies the following two properties.

- Each partition is a shortest path (we will be showing in Theorem 1 that any geodesic that lies inside a peel is a shortest path).
- We will show that the number of peels crossed by any finite geodesic is countable. Hence, the number of partitions in  $xa$  is also countable.

The number of peels crossed by any finite geodesic  $xa$  is countable for the following reason. Since  $xa$  is a geodesic, there is no polyhedral vertex in the interior of  $xa$ . Thus, the intersection of a peel with  $xa$  is either null or of non-zero length. Hence, we can order the peels that intersect  $xa$  in the direction from  $x$  to  $a$ . Thus, the set of peels that have a non-empty intersection with  $xa$  is countable.

Let  $b_i$  denote the image of  $b$  in  $source(a_i)$  for each  $i, 0 \leq m$ . Consider the three points  $a_i, a_{i+1}, a_{i+2}$  for any  $i, 0 \leq i \leq m - 2$ . We have already mentioned that



**Fig. 8.**  $source(ab) \geq SP(ab)$  in extended source unfolding: Case 2

$b_i, b_{i+1}$  and  $b_{i+2}$  denotes the images of the point  $b$  in the  $source(a_i)$ ,  $source(a_{i+1})$  and  $source(a_{i+2})$ , respectively. We have the following analogy to Case 1. We can view the four points  $a_i, a_{i+1}, a_{i+2}$  and  $b_i$  as  $x, y, a$  and  $b$  in Case 1, respectively. An argument on lines similar to that of Case 1 shows the following (see Fig.8).

Since  $a_i b_i$  and  $a_i a_{i+1}$  are shortest paths in  $source(a_i)$ ,  $a_{i+1} b_i \geq SP(a_{i+1} b_i) = SP(a_{i+1} b_{i+1}) = SP(a_{i+1} b)$ . Thus,  $b_{i+1}$  lies on the disc  $C(a_{i+1}, a_{i+1}, b_i)$ .

Since  $a_{i+1} a_i$  and  $a_{i+1} b_{i+1}$  are shortest paths in  $source(a_{i+1})$ ,  $a_i b_{i+1} \geq a_i b_i = SP(a_i b_i)$ . Hence,  $b_{i+1}$  does not lie inside the disc  $C(a_i, a_i, b_i)$ .

Thus,  $b_{i+1}$  lies on the disc  $C(a_{i+1}, a_{i+1}, b_i)$  but not inside the disc  $C(a_i, a_i, b_i)$ . That is,  $b_{i+1}$  lies on the portion of the disc  $C(a_{i+1}, a_{i+1}, b_i)$  that is on or to the right of  $L_i$  where  $L_i$  is the normal from  $b_i$  to the x-axis.

Since, for each  $j \geq i + 2$ , the point  $a_j$  lies to the right of  $a_{i+1}$ , the above mentioned portion of the disc  $C(a_{i+1}, a_{i+1}, b_i)$  lies inside  $C(a_j, a_j, b_i)$ . Thus we have  $a_j b_i \geq a_j b_{i+1}$  for each  $j \geq i + 2$ . In particular we have  $a_m b_i \geq a_m b_{i+1}$ .

Since the above claims are true for each  $i, 0 \leq i \leq m - 2$ , we get

$$a_m b_0 \geq a_m b_1 \geq a_m b_2 \geq a_m b_3 \geq \dots \geq a_m b_{m-1}.$$

Since  $a_{m-1} a_m$  and  $a_{m-1} b_{m-1}$  are shortest paths in the Source Unfolding  $source(a_{m-1})$ , we have  $a_m b_{m-1} \geq SP(a_m b)$ . Thus we have  $source(ab) = a_m b_0 \geq SP(a_m b) = SP(ab)$  (recall  $a_m = a$ ), and we have proved the theorem for case 2.  $\square$ .

## 7 A Necessary Condition for Short-cuts in a Planar Unfolding

Our result on Extended Source Unfoldings above provides a necessary condition for  $ab$  to be a short-cut on a Planar Unfolding, i.e., for  $planar(ab) \geq SP(ab)$ .

**Theorem 6.** *Consider three points  $x, a, b$  on some Planar Unfolding such that  $xa$  and  $xb$  are images of geodesics completely within this unfolding, and further, the  $xa$  geodesic is actually a shortest path on  $\mathcal{P}$ . Then  $\text{planar}(ab) \geq SP(ab)$ , and therefore  $ab$  cannot be a short-cut.*

**Proof.** Since the neighborhoods of  $x$  in the Source and the Planar Unfoldings are identical, the angle between  $xa$  and  $xb$  in these two unfoldings are the same. Hence, we have  $\text{planar}(ab) = \text{source}(ab)$ . However, according Theorem 5, we have  $\text{source}(ab) \geq SP(ab)$ . Hence, we have  $\text{planar}(ab) = \text{source}(ab) \geq SP(ab)$ .  
□

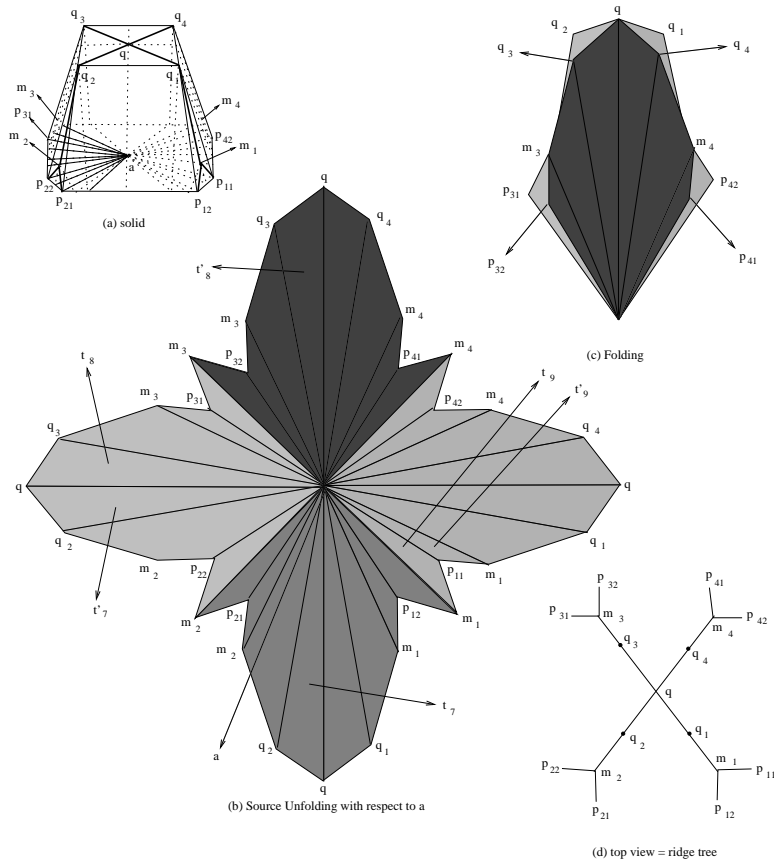
In particular note that if  $a$  lies on an edge of  $\mathcal{P}$  and we take  $x$  to be on the same edge, if  $xa$  is a geodesic then  $ab$  cannot be a short-cut. Since geodesics are locally verifiable in contrast to shortest paths, this is a non-trivial condition. A simple further corollary is that any planar unfolding between edges  $e_0$  and  $e_k$  with a geodesic from  $e_0$  to  $e_k$  which lies completely inside this planar unfolding cannot have shortcuts which cross this geodesic.

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## Appendix

### 8 Compacted Source Unfoldings and Proofs of Properties



**Fig. 9.** Compacted Source Unfoldings

Let  $source(a)$  denote the Source Unfolding of a convex polyhedron  $\mathcal{P}$  with respect to  $a$ . Recall from the definition of the Source Unfolding that each point on the boundary of the Source Unfolding (except possibly the polyhedral vertices) has more than one shortest path from  $a$  on the polyhedron. Such a point is called a *ridge point* with respect to the point  $a$  (see [7]). The set of all ridge points with respect to the point  $a$  on the convex polyhedron constitutes a tree on the polyhedron called the *ridge tree* with respect to  $a$  or equivalently, the *ridge tree* with respect to  $source(a)$ . The ridge tree has *ridge edges* and *ridge*

*nodes*. The ridge edges are geodesics on the polyhedron. Thus, each point on the interior of a ridge edge in the ridge tree is a point in the interior of a face or an edge in the polyhedron as well. Further, each point on the interior of a ridge edge has exactly two shortest paths from  $a$ . Each polyhedral vertex is a ridge node in the ridge tree. However, not all ridge nodes are polyhedral vertices. If a ridge node is not a polyhedral vertex, then it is a point on a face or an edge of the polyhedron where three or more shortest paths from  $a$  will meet. See Fig.9(a) and (b) for an example. The ridge tree is shown in bold in this figure. The polyhedral vertices  $q_i, 1 \leq i \leq 4$ , are internal nodes in the ridge tree whereas the polyhedral vertices  $p_{ij}, 1 \leq i \leq 4$  and  $1 \leq j \leq 2$  are the leaves. The ridge nodes  $m_i, 1 \leq i \leq 4$ , are internal points on the faces of the polyhedron each having three shortest paths from  $a$ . The ridge node  $q$  lies on a face of the polyhedron with four shortest paths from  $a$ .

Let us assume that the  $source(a)$  is triangulated so that each triangle has a ridge edge as its base and  $a$  as its vertex. We call these triangles as *ridge triangles*. Note that each ridge edge is shared by exactly two ridge triangles that are similar. We say that these two similar triangles are *images* of each other. Suppose that we paste (to be specific, superimpose) each ridge triangle in the  $source(a)$  to its image so that the corresponding vertices, edges and internal points coincide. We call the resulting structure as the *Compacted Source Unfolding*. Intuitively, the Source Unfolding folds like an umbrella to form its compact version. See Fig.9 for an example. We have shown the ridge tree with respect to point  $a$  in bold lines in Fig.9(a) and the corresponding Source unfolding  $source(a)$  in Fig.9(b). The ridge triangles that share the ridge edges are pasted together in Fig.9(c) to get the Compacted Source. The top view of this is shown in Fig.9(d); this view is identical to the ridge tree itself.

Let  $\rho$  be the ridge tree that corresponds to  $source(a)$ . Choose any pair of maximal shortest paths  $SP(au)$  and  $SP(av)$  in  $source(a)$ . As we defined earlier, the  $source(a_{(uv)})$  is the portion of  $source(a)$  clockwise from  $SP(au)$  to  $SP(av)$ . Since  $SP(au)$  and  $SP(av)$  are maximal shortest paths in  $source(a)$ ,  $u$  and  $v$  lie on the ridge tree  $\rho$ . Let  $\rho_{uv}$  denote the path on the ridge tree from  $u$  to  $v$ . Let  $\tau_{uv}$  denote the sequence of ridge triangles in the  $source(a_{(uv)})$ . Among the ridge triangles in  $\tau_{uv}$ , some of the pairs can be mutually the images of each other. Let  $\tau'_{uv} = t_1 \dots t_k$  denote the subsequence of  $\tau_{uv}$  that is obtained after removing all pair of triangles in  $\tau_{uv}$  that are mutually the images of each other. We call  $\tau'_{uv}$  as the *back-bone* of  $source(a_{(uv)})$ .

See Fig.10. We have the same polyhedron as in Fig.9. We have chosen  $q_4 = u$  and  $p_{12} = v$ . The  $source(a_{(uv)})$  is the clockwise portion from  $aq_4$  to  $ap_{12}$ . The back-bone will have the triangles  $\Delta aq_4q$ ,  $\Delta aqq_1$ ,  $\Delta aq_1m_1$ ,  $\Delta am_1p_{11}$ ,  $\Delta ap_{11}m_1$  and  $\Delta am_1p_{12}$ . We have labelled these triangles  $t_1$  through  $t_6$  respectively. The back-bone is shown by a curve in bold in Fig.10(c). The back-bone consists of the triangles  $t_1, t_2, t_3, t_6$ .

Consider the back-bone formed by the sequence of triangles  $t_1 \dots t_k$ . For those pairs of triangles  $t_{i-1}$  and  $t_i$  that have the same shortest path (on the polyhedron)  $an_i$  as their boundary, these two triangles are adjacent in a nat-

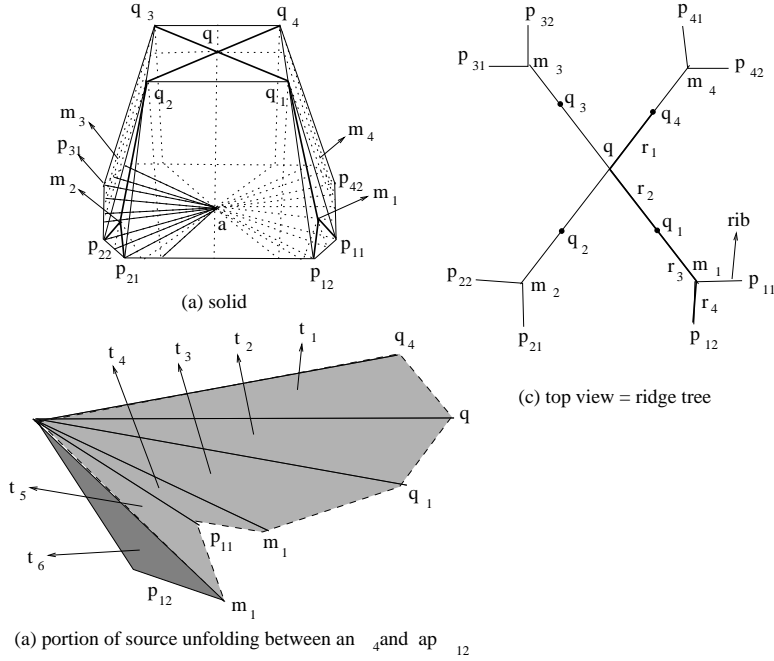


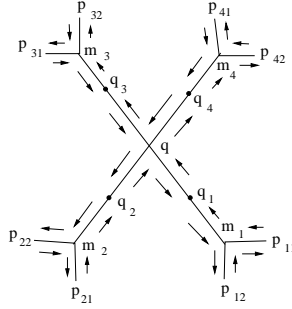
Fig. 10. Back-bone and ribs

ural way in the back-bone because they are neighbors in the polyhedron also. Whereas, those pairs of triangles  $t_{i-1}$  and  $t_i$  that have different shortest paths  $an_i$  as their boundary, a portion of the Compact Source will also be attached to their *now* common boundary  $an_i$ . This portion consists of ridge triangles that are interposed between  $t_{i-1}$  and  $t_i$  in  $\tau_{uv}$  (or equivalently,  $source(a_{(uv)})$ ). We call this portion of the Compact Source that also belongs to  $source(a_{(uv)})$  and that is not part of the back-bone, but attached at  $an_i$  to the backbone, as a *rib* with respect to  $source(a_{(uv)})$ .

Consider the four triangles  $t_3, t_4, t_5$  and  $t_6$  that share the common edge  $am_1$  in the Compact Source (see Fig.10). The triangles  $t_3$  and  $t_6$  are adjacent in the back-bone of  $source(a_{(uv)})$ , but their common boundary  $am_1$  in the back-bone is not the same shortest path on the polyhedron. The two triangles  $t_4$  and  $t_5$  being mutually the images of each other, get pasted back to back constitute the rib attached to the back-bone along  $am_1$  between  $t_3$  and  $t_6$ .

Consider the sequence  $\tau'_{uv} = t_1 \dots t_k$ . Each ridge triangle has a ridge edge  $r_i$  in its boundary. Also note that  $r_i$  and  $r_{i+1}$  share the ridge vertex  $n_i$  for each triangle  $t_i$ . Let  $n_0 = u, n_k = v$ . Thus we have the sequence of ridge edges  $r_1 \dots r_k$  corresponding to  $\tau'_{uv} = t_1 \dots t_k$ . The following lemma proves the relation between  $r_1 \dots r_k$  and  $\rho_{uv}$ .

**Lemma 4.**  $\rho_{uv} = r_1 \dots r_k$ .



**Fig. 11.** depth-first traversal

**Proof.** Without loss of generality, we shall assume that the ridge tree is rooted at  $u$ . Our proof involves the following two claims.

- The sequence of ridge edges in the boundary of  $source(a)$  (and hence, the ridge triangles in  $source(a)$ ) have a one to one correspondence to the complete tree traversal of the ridge tree.
- The sequence of ridge edges in the boundary of  $source(a_{(uv)})$  (and hence, the ridge triangles in  $source(a_{(uv)})$ ) have a one to one correspondence to the portion from  $u$  to  $v$  of the tree traversal of the ridge tree that is rooted at  $u$ .
- The removal of duplicated ridge triangles in  $source(a_{(uv)})$  is equivalent to removal of the duplicated edge traversals while traversing the ridge tree from  $u$  to  $v$ .
- The sequence  $\tau'_{uv}$  of ridge triangles that are left over in  $\tau_{uv}$  after removing the duplicates is the same as the path in the ridge tree from  $u$  to  $v$ , that is  $\rho_{uv}$ .

Thus we have  $\rho_{uv} = r_1 \dots r_k$ .  $\square$

**Lemma 5.** *The back-bone of  $source(a_{(uv)})$  is a convex polygon.*

*Proof.* Let  $\tau'_{uv} = t_1 \dots t_k$  be the sequence of ridge triangles in the back-bone of  $source(a_{(uv)})$  as defined in the Lemma 4. We claim that the back-bone is a convex polygon for the following four reasons.

- Adjacent triangles  $t_i$  and  $t_{i+1}$  do not intersect except along their common side  $an_i$ .
- The angle at  $a$  is less than  $\pi$  for the following reason. Folding consists of only the triangles  $t_1, t_2, \dots, t_k$  and not their images  $t'_1, t'_2, \dots, t'_k$ . Each  $t_i$  that is incident at  $a$  has its image  $t'_i$  also incident at  $a$ , and  $t_i$  and  $t'_i$  are similar triangles for each  $i$ . Since the angle at  $a$  is at most  $2\pi$  in a convex polyhedron, the angle sum at  $a$  of the triangles  $t_1, t_2, \dots, t_k$  is at most  $\pi$ .

– The vertices of the Compact Source are  $a, n_0 = u, n_1, n_2, \dots, n_k = v$  as defined above. Since  $t_1$  and  $t_k$  are triangles, the internal angles at  $n_0$  and  $n_k$  are less than  $\pi$ . Consider any other  $n_i, 0 < i < k$ . Each such  $n_i$  is a ridge node. The internal angle at  $n_i$  in the back-bone is the sum of the internal angles at  $n_i$  in the triangles  $t_i$  and  $t_{i+1}$ . We will show in the next paragraph that the four triangles  $t_i, t'_i, t_{i+1}$  and  $t'_{i+1}$  that share the vertex  $n_i$  can together make an angle  $< 2\pi$  at  $n_i$ . Since the sum of the angles at  $n_i$  in  $t_i$  and  $t_{i+1}$  is identical to that in  $t'_i$  and  $t'_{i+1}$ , the angle at each  $n_i$  is less than  $\pi$  in the back-bone. We will now show that four triangles  $t_i, t'_i, t_{i+1}$  and  $t'_{i+1}$  can together make an angle  $< 2\pi$  at  $n_i$ . We have the following two cases.

- $n_i$  is not a polyhedral vertex: Then there are at least three shortest paths to that  $n_i$  from  $a$ . Hence, there are at least six triangles incident at  $n_i$ . Thus the four triangles  $t_i, t'_i, t_{i+1}$  and  $t'_{i+1}$  that share the vertex  $n_i$  can together make an angle  $< 2\pi$  at  $n_i$ .
- $n_i$  is a polyhedral vertex: Then due to the curvature at  $n_i$ , the total angle at  $n_i$  is  $< 2\pi$ . Thus, the four triangles  $t_i, t'_i, t_{i+1}$  and  $t'_{i+1}$  that share the vertex  $n_i$  can together make an angle  $< 2\pi$  at  $n_i$ .

Since the back-bone is a simple polygon with each internal angle  $< \pi$ , it is convex.

### Proof of Theorem 1.

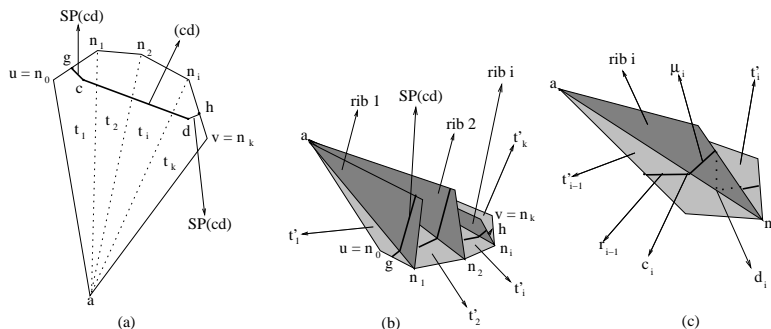


Fig. 12. Geodesics on a peel are unique shortest paths

See Fig.12(a). Let  $c$  and  $d$  be any pair of points on a peel  $p_1$  in  $source(a)$  of the polyhedron  $P$ . Since a peel is a convex polygon (see [7] [10]), there exists a geodesic  $(cd)$  joining  $c$  and  $d$  that lies entirely on the peel  $p_1$ . Contrary to the lemma, suppose that this geodesic  $(cd)$  is not a shortest path. Then we have the following two cases to consider.

**Case 1: The polyhedron is degenerate with only three vertices  $a, c$  and  $d$ :** In this case we have just two peels  $p_1$  and  $p_2$  in the polyhedron. The peel  $p_1$

consists of exactly one ridge triangle  $t_1$  whose vertices are  $a$ ,  $c$  and  $d$ . The image  $t'_1$  of  $t_1$  is the peel  $p_2$  itself with the same three vertices. Thus we have a unique path from  $c$  to  $d$  (that is a ridge edge as well) that is the boundary between the peels  $p_1$  and  $p_2$ .

**Case 2: Any other situation:** Since we have assumed that  $(cd)$  is not a shortest path, we should have a shortest path  $SP(cd)$  between  $c$  and  $d$  that is shorter than  $(cd)$  and that does not lie entirely on the peel  $p_1$ . Let  $u$  and  $v$  be the vertices of the polyhedron that lie on the peel  $p_1$ . Let  $t_1 \dots t_k$  be the sequence of ridge triangles in the peel  $p_1$  (see Fig.12(a)).

Consider the  $source(a_{(uv)})$ , its back-bone and ribs. The path in the ridge tree between  $u$  and  $v$  is the boundary of the peel  $p_1$  between the two vertices  $u$  and  $v$ . Thus, the back-bone of  $source(a_{(uv)})$  is formed by the sequence  $t_1 \dots t_k$  with their images pasted on their back. Let  $\{n_0 = u, n_1, n_2, \dots, n_{s-1}, n_k = v\}$  be the sequence of ridge nodes in the ridge tree between  $u$  and  $v$  where  $n_{i-1}n_i$  is a ridge edge  $r_i$  for each  $i$ . By definition each  $n_i, 0 < i < k$ , is a ridge node that is not a polyhedral vertex. Thus, there are at least six ridge triangles incident at each  $n_i, 0 < i < k$ . Since the ridge triangles  $t_{i-1}$  and  $t_i$  are adjacent along the edge  $an_i$  on the peel  $p_1$  for each  $i, 0 < i < k$ , the triangles  $t'_{i-1}$  and  $t'_i$  cannot have a common edge  $an_i$  in the polyhedron. Thus, we have a rib (call it rib  $i$ ) attached to the back-bone between the images  $t'_i$  and  $t'_{i+1}$  for each  $i, 1 \leq i < k$  in the Compact Source of  $source(a_{(uv)})$  (see Fig.12(b)).

Since we have assumed that the shortest path  $SP(cd)$  does not lie on the peel  $p_1$ , it should lie on multiple peels. Thus, we can partition  $SP(cd)$  into maximal connected components such that each maximal connected component lies on a single peel in the Source Unfolding. These components are of the following types.

- *Components on  $t_1 \dots t_k$  or equivalently, on  $source(a_{(uv)}) = p_1$ :* This includes the first connected component  $cg$  and the last connected component  $hd$  on the peel  $p_1$  and possibly other intermediate components if any.
- *One or more connected components on  $t'_1 \dots t'_k$  or equivalently, on the back-bone of  $source(a_{(uv)})$  which is the compliment of  $source(a_{(uv)}) = p_1$ .*
- *One or more connected components on the ribs of  $source(a_{(uv)})$ .*

Our proof involves the following major steps. We will replace  $SP(cd)$  by a continuous curve  $C$  that lies entirely on the peel  $p_1$  and that is *not* longer than  $SP(cd)$ , that is we will have  $C \leq SP(cd) < (cd)$  (the latter inequality is our assumption at the beginning of this proof). However, any continuous curve on the peel  $p_1$  from  $c$  to  $d$  cannot be shorter than the straight line segment  $(cd)$ . Therefore, we have the straight line segment  $(cd)$  on the peel  $p_1$  shorter than  $C$ , that is  $(cd) < C$ . But, this contradicts our claim that we have  $C \leq SP(cd) < (cd)$ . Thus, we conclude that  $(cd)$  is the unique shortest path.

It is now sufficient to shown that we get a curve  $C$  that lies entirely on the peel  $p_1$  and that is not longer that  $SP(cd)$ . Thus, it is done in the following steps.

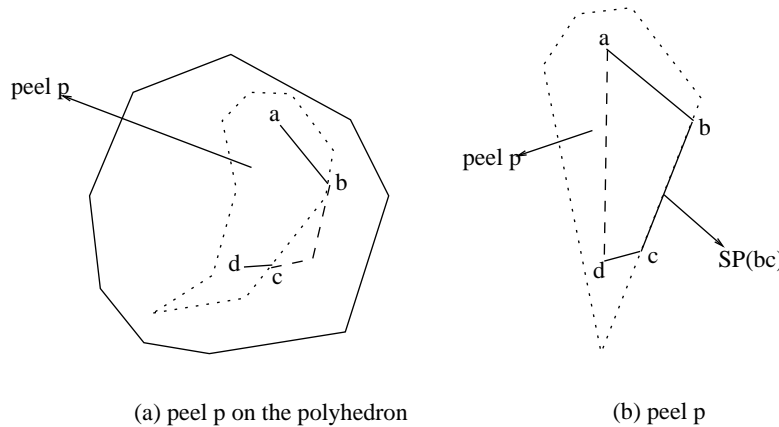
- **Step 1:** We retain the connected components on  $t_1 \dots t_k$  as they are. Hence, this step does not contribute to any change in the length of the resulting curve (see Fig.12(a)).

- We replace the connected component of  $SP(cd)$  on each  $t'_i$  by an identical curve (of the same length) on its image  $t_i$ . This is possible because  $t'_i$  and  $t_i$  are identical. Hence, this step does not contribute to any change in the length of the resulting curve (see Fig.12(b)).
- Let  $\mu_i$  denote the portion of  $SP(cd)$  that lies on a rib between  $t'_{i-1}$  and  $t'_i$  for each  $i, 1 < i < k$  (see Fig.12(c)). Let  $c_i$  and  $d_i$  be the end points of  $\mu_i$ . Then  $c_i$  lies on the boundary  $an_i$  of  $t'_{i-1}$ . Similarly,  $d_i$  lies on the boundary  $an_i$  of  $t'_i$ . Thus, both  $c_i$  and  $d_i$  lie on the common edge  $an_i$  in the Compact Source that is shared by the ridge triangles  $t_{i-1}, t_i, t'_{i-1}$  and  $t'_i$  in addition to the ridge triangles that are in the rib  $i$  that contains  $\mu_i$ . By triangle inequality,  $\mu_i$  is longer than the straight line segment  $c_i d_i$  that is a portion of  $an_i$ . Let us replace  $\mu_i$  by the straight line segment  $(c_i d_i)$  that is a portion of  $an_i$ . Since  $(c_i d_i)$  lies on the common boundary  $an_i$  in the Compact Source, we can view  $(c_i d_i)$  as a straight line segment on the peel  $p_1$  itself. Hence, this step reduces the length of the resulting curve.

Note that the resulting curve  $C$  after the above substitutions on  $SP(cd)$  is continuous and lies entirely on the peel  $p_1$ , and it is not longer than  $SP(cd)$ .  $\square$

Theorem 1 has the following consequences.

**Proof of Corollary 1.**

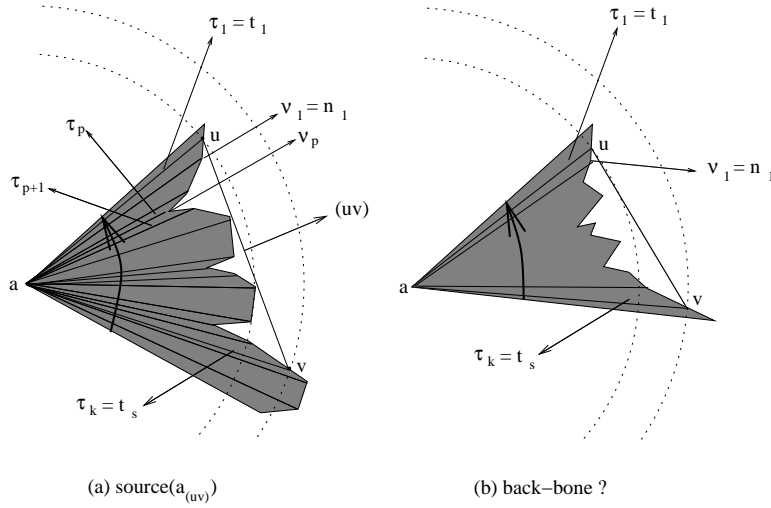


**Fig. 13.** Intersection of a shortest path with a peel

Contrary to the lemma, suppose that the intersection of a shortest path  $S$  with a peel  $p$  is not connected. Without loss of generality we assume that  $ab$  and  $cd$  are maximal connected components of the shortest path  $S$  that lie on the peel  $p$  and the portion  $bc$  of  $S$  lies outside the peel  $p$  as shown in Fig.13(a). The portion  $bc$  of  $S$  is a shortest path because  $S$  itself is a shortest path. However, according to Theorem 1, since  $b$  and  $c$  are on the same peel  $p$ , there is a unique

shortest path  $bc$  between them on the same peel  $p$  as shown in Fig.13(b) and hence, we cannot have one more shortest path that is a portion of  $S$  outside the peel  $S$ , a contradiction. Hence, the intersection of a shortest path with a peel is connected.  $\square$

**Proof of Theorem 2.**



**Fig. 14.**  $source(a_{(uv)})$  and its back-bone

Contrary to the theorem, suppose that there is no shortest path  $SP(au)$  on any one of the ridge triangles in the sequence  $\tau_2 \dots \tau_k$ . Then we will show that the back-bone of  $source(a_{(uv)})$  is not a convex polygon. However, this contradicts Lemma 5, according to which back-bone of a  $source(a_{(uv)})$  for any choice of  $a$ ,  $u$  and  $v$  is a convex polygon. Thus, the theorem holds true. Now we will begin with an assumption that we do not have a shortest path  $SP(au)$  on any one of the ridge triangles in the sequence  $\tau_2 \dots \tau_k$  and show that the back-bone of  $source(a_{(uv)})$  is not a convex polygon.

Recall how we got the back-bone of  $source(a_{(uv)})$ . Since  $u$  and  $v$  are ridge points, we have a unique path  $\rho_{uv}$  from  $u$  to  $v$  on the ridge tree.  $\rho_{uv}$  contains a sequence of line segments  $r_1 \dots r_s$  where all except possibly  $r_1$  and  $r_s$  are complete ridge edges while  $r_1$  and  $r_s$  are portions of ridge edges ending at  $u$  and  $v$  respectively. We continue to call each line segment in  $r_1 \dots r_s$  a ridge edge for the sake of convenience. Let  $t_1 \dots t_s$  denote the sequence of triangles that corresponds to the sequence  $r_1 \dots r_s$  such that  $r_i$  is one of the sides of the triangle  $t_i$  for each  $i$ . We continue to call each triangle in the sequence  $t_1 \dots t_s$  a ridge triangle for the sake of convenience. The union of the triangles in the sequence  $t_1 \dots t_s$  is the back-bone of  $source(a_{(uv)})$ .

Let  $\nu_i$  be the ridge node shared between the ridge triangles  $\tau_i$  and  $\tau_{i+1}$  for each  $i, 1 \leq i < k$  in the  $source(a)$ . Let  $n_i$  be the ridge node shared between the ridge triangles  $t_i$  and  $t_{i+1}$  for each  $i, 1 \leq i < s$ , in the back-bone.

Our proof involves the following major claims (see Fig.14 (a) and (b)).

- **Claim 1:** The ridge node  $\nu_1$  in  $source(a)$  is the same as the ridge node  $n_1$  in the back-bone. Further,  $n_1$  is a polygonal vertex of the back-bone (back-bone is a polygon).
- **Claim 2:** The points  $a$ ,  $u$ , and  $v$  are polygonal vertices of the back-bone (back-bone is a polygon).
- **Claim 3:** The vertex  $n_1$  in the back-bone lies inside the triangle formed by the vertices  $a$ ,  $u$ , and  $v$  in the back-bone. A convex polygon cannot have one of its vertices inside the triangle formed by any other three vertices. Thus, the back-bone is not convex.

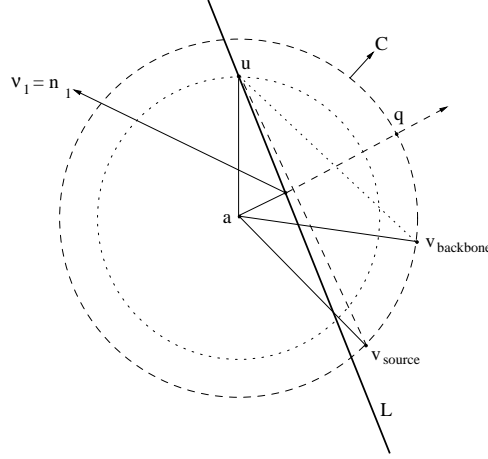
We will now prove these three claims.

**Claim 1:**  $\nu_1 = n_1$ : We have assumed at the beginning of this proof that we do not have a shortest path  $SP(av)$  on any one of the ridge triangles in the sequence  $\tau_2 \dots \tau_k$ . Thus, none of the triangle in the sequence  $\tau_2 \dots \tau_k$  is an image of  $\tau_1$ . Thus, the back-bone will have  $\tau_1$  as its first triangle, that is  $\tau_1 = t_1$ . Since,  $\nu_1$  is one of the vertices of  $\tau_1$ , we have  $\nu_1 = n_1$  as a vertex of the back-bone (viewed as a polygon).

**Claim 2:**  $a$ ,  $u$ , and  $v$  are polygonal vertices in the back-bone: Since  $\tau_1 = t_1$ , the vertices  $a$  and  $u$  of  $\tau_1$  will continue to be vertices in the back-bone. Since we have assumed that  $SP(av) \geq SP(au)$ , the  $\angle vau < \pi$  and  $uv$  is outside  $source(a)$  everywhere except at its end points, there does not exist a point  $q$  on  $source(a_{(uv)})$  such that  $SP(aq) \geq SP(av)$ . Thus, we cannot have another shortest path  $SP(av)$  in  $source(a_{(uv)})$ . Hence, we cannot have the image of  $\tau_k$  in the sequence  $\tau_1 \dots \tau_{k-1}$  as well. Thus,  $\tau_k$  belongs to the back-bone as the last triangle, that is we have  $\tau_k = t_s$ . Thus, the vertex  $v$  of the ridge triangle  $\tau_k$  is a vertex in the back-bone as well. Thus  $a$ ,  $u$ , and  $v$  are polygonal vertices in the back-bone. Let us denote the triangles formed by the three points  $v$ ,  $a$  and  $u$  in  $source(a_{(uv)})$  and its back-bone as  $\Delta vau_{(source)}$  and  $\Delta vau_{(backbone)}$  respectively.

**Claim 3:**  $n_1$  is inside the triangle  $\Delta vau_{(backbone)}$ : Let  $\angle vau_{(source)}$  and  $\angle vau_{(backbone)}$  denote the angle at the vertex  $a$  in the triangles  $\Delta vau_{(source)}$  and  $\Delta vau_{(backbone)}$  respectively. Then  $\angle vau_{(source)}$  is the sum of the angles at  $a$  in the triangles  $\tau_1 \dots \tau_k$ , whereas  $\angle vau_{(backbone)}$  is the sum of the angles at  $a$  in the triangles  $t_1 \dots t_s$ . We will show in the last paragraph that  $t_1 \dots t_s$  is a proper subsequence of  $\tau_1 \dots \tau_k$ . Thus, we have  $\angle vau_{(backbone)} < \angle vau_{(source)}$ . Since  $\tau_1 = t_1$  is the first triangle in each of the structures  $source(a_{(uv)})$  and its back-bone, its contribution to each of the angles  $\angle vau_{(backbone)}$  and  $\angle vau_{(source)}$  is the same. Therefore we can compare the contributions of the rest of the triangles to get the relation  $\angle van_1_{(backbone)} < \angle van_1_{(source)}$ . Note that  $\angle van_1_{(backbone)} > 0$  because the triangle  $\tau_k = t_s$  contributes to this angle. We will show in the next paragraph that  $n_1$  lies inside the triangle  $\Delta vau_{(backbone)}$  because the following four conditions are satisfied.

- $\angle vau_{1(source)} > \angle van_{1(backbone)} > 0$ ,
- $SP(av) \geq SP(au)$ ,
- $\tau_1 = t_1$  and
- $SP(av)$  is the same in both the  $source(a_{(uv)})$  and its back-bone.



**Fig. 15.** super-imposition of  $source(a_{(uv)})$  and its back-bone

Let us superimpose the  $source(a_{(uv)})$  and its back-bone so that the triangles  $\tau_1 = \Delta auv_1$  and  $t_1 = \Delta aun_1$  coincide (see Fig.15). This is possible because  $\tau_1 = t_1$ . We shall call the point  $v$  in the  $source(a_{(uv)})$  and its back-bone as  $v_{(source)}$  and  $v_{(backbone)}$  respectively. We draw the circle  $C$  with center at  $a$  and radius  $SP(av)$ . Both  $v_{(source)}$  and  $v_{(backbone)}$  lie on the circle  $C$  because  $SP(av)$  is the same in both the  $source(a_{(uv)})$  and its back-bone. Extend the line segment  $an_1$  to meet the circle  $C$  at  $q$ . Let  $C'$  denote the portion of the circle  $C$  that we encounter when we move anti-clockwise from  $v_{(source)}$  to  $q$ . Since we have  $\angle vau_{1(source)} > \angle van_{1(backbone)} > 0$ , we can have  $v_{(backbone)}$  only on the portion  $C'$  of the circle  $C$ . Since we have assumed that  $SP(av) \geq SP(au)$ , there does not exist a point  $w$  that lies inside the triangle  $\Delta vau$  such that  $aw = SP(av)$ . Thus for each possible position for  $v_{(backbone)}$  on  $C'$ , we have  $n_1$  inside the triangle  $\Delta vau_{(backbone)}$  as well.

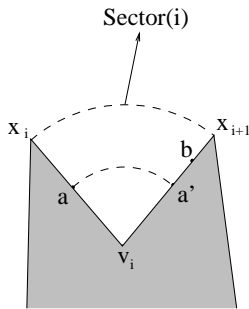
$t_1 \dots t_s$  is a proper subsequence of  $\tau_1 \dots \tau_k$ : According to the definition of back-bone,  $t_1 \dots t_s$  is a subsequence of  $\tau_1 \dots \tau_k$ . Thus, it is sufficient to show that some of the ridge triangles in  $\tau_1 \dots \tau_k$  do not belong to  $t_1 \dots t_s$ , so that  $t_1 \dots t_s$  is a proper subsequence of  $\tau_1 \dots \tau_k$ . Consider the two ridge triangles  $\tau_1$  and  $\tau_k$ . We claim that  $\tau_1$  and  $\tau_k$  belong to two different peels for the following reason. Contrary to the claim, suppose that  $\tau_1$  and  $\tau_k$  belong to the same peel. Then we will have both the points  $u$  and  $v$  on this peel and hence, the line

segment  $(uv)$  lies on this peel. This contradicts our assumption that  $(uv)$  lies outside  $source(a)$  except for its end points. Hence, we conclude that  $\tau_1$  and  $\tau_k$  lie on different peels. Thus, at least one of the ridge nodes say an  $\nu_p, 1 \leq p < k$ , is a polyhedral vertex. Then the two ridge triangles  $\tau_p$  and  $\tau_{p+1}$  are images of each other, and hence, they belong to a rib of  $source(a_{(uv)})$ . Hence,  $\tau_p$  and  $\tau_{p+1}$  belong to  $\tau_1 \dots \tau_k$  but not to  $t_1 \dots t_s$ . Hence,  $t_1 \dots t_s$  is a proper subsequence of  $\tau_1 \dots \tau_k$ .  $\square$

## 9 Star Unfoldings: Proof of Theorem 3

**Proof.** There are three cases, depending upon whether  $ab$  lies completely inside the Star Unfolding, completely outside the Star Unfolding, or partly inside and partly outside the Star Unfolding.

**Case 1.** If  $ab$  lies completely inside the Star Unfolding then, by Fact 1, this line segment is an image of a geodesic on the surface of  $\mathcal{P}$  (see Fig.1(b)). The length of this geodesic is clearly at least  $SP(ab)$ . Theorem 3 holds, therefore, for this case.

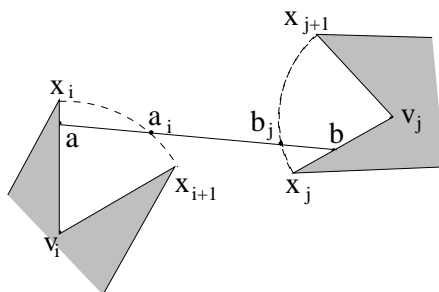


**Fig. 16.** Case 2: Both  $a$  and  $b$  on the same sector

**Case 2.** Next, suppose  $ab$  lies completely outside the Star Unfolding. Clearly, both  $a$  and  $b$  must lie on the boundary of the Star Unfolding. Recall that, by Lemma 1, the Star Unfolding is a simple polygon. Let  $Sector(i)$  and  $Sector(j)$  be sectors containing  $a$  and  $b$ , respectively. There are two sub cases in this case, depending upon whether or not  $i = j$ .

First, suppose  $i = j$ . Then it is easily seen that  $ab$  is completely contained within  $Sector(i)$  (see Fig.16). Recall that  $Sector(i)$  is a sector of the circle centered at  $v_i$  and is bounded by the two radii  $x_i v_i$  and  $x_{i+1} v_i$ . Since, by Lemma 1, this sector is completely outside the Star Unfolding, both  $a$  and  $b$  lie on the

boundary of the  $Sector(i)$ . Since the boundary of the  $Sector(i)$  consists of the two radii  $x_i v_i$  and  $x_{i+1} v_i$ , and  $ab$  is assumed to be outside the Star Unfolding, we assume without loss of generality that  $a$  must be on  $x_i v_i$  and  $b$  on  $x_{i+1} v_i$ . Further, without loss of generality,  $star(v_i a) \leq star(v_i b)$ . Let  $a'$  be the unique point on  $x_{i+1} v_i$  such that  $a$  and  $a'$  are images of the same point on  $\mathcal{P}$ . We claim that  $a'b < ab$ . Consider the two triangles  $\Delta av_i b$  and  $\Delta a'v_i b$ . The angle  $\theta = \angle av_i b > \angle a'v_i b = 0$ , where  $\theta$  is the curvature of the polyhedron at  $v_i$ . Further,  $v_i a = v_i a'$  and  $v_i b$  is common to both the triangles. Hence, we have  $a'b < ab$ . That is, we have  $star(a'b) \leq star(ab)$ . Since  $a'b$  is completely inside the Star Unfolding, by Case 1, it follows that  $SP(a'b) \leq star(a'b)$ . Finally, since  $a'$  and  $a$  correspond to the same point on  $\mathcal{P}$ , we have  $SP(ab) \leq SP(a'b)$ . Thus we have  $SP(ab) \leq SP(a'b) \leq star(a'b) \leq star(ab)$ , and the theorem follows in this case.



**Fig. 17.** Case 2: Both  $a$  and  $b$  on the same sector

Second, suppose  $i \neq j$  (Fig.17). Note that  $a$  must lie on one of  $v_i x_i$  or  $v_i x_{i+1}$ ; similarly,  $b$  must lie on one of  $v_j x_j$  or  $v_j x_{j+1}$ . Without loss of generality, we assume that  $a$  lies on  $v_i x_i$  and  $b$  on  $v_j x_j$  (the other cases are identical). We show in the next paragraph that  $star(ax_i) + star(x_j b) \leq star(ab)$ . Note that  $x_i$  and  $x_j$  are images of the same point  $x$  on  $\mathcal{P}$ . Then, since  $ax_i, x_j b$  are both in the interior of the Star Unfolding, using Case 1, there exists a path from  $a$  to  $b$  through  $x$  on  $\mathcal{P}$  which is no longer than  $star(ax_i) + star(x_j b) \leq star(ab)$ . It follows that  $star(ab) \geq SP(ab)$ .

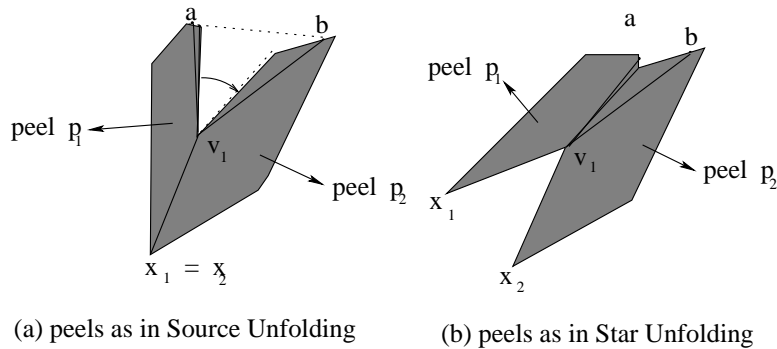
We need to show that  $star(ax_i) + star(x_j b) \leq star(ab)$ . Recall from Lemma 1 that  $Sector(i)$  and  $Sector(j)$  do not overlap. Let  $aa_i$  be the maximal portion of  $ab$  within  $Sector(i)$  and let  $b_j b$  be the maximal portion of  $ab$  within  $Sector(j)$ ; then  $aa_i$  and  $b_j b$  are disjoint line subsegments of  $ab$ . So  $star(aa_i) + star(b_j b) \leq star(ab)$ . Further,  $star(aa_i) \geq star(ax_i)$  and  $star(b_j b) > star(x_j b)$  (from any point inside a circle, the nearest point on the circumference is the one which lies on the straight line joining this point to the center). The claim follows.

**Case 3.** Suppose  $ab$  doesn't lie completely in the interior or completely in the exterior of the Star Unfolding. In this case, we decompose  $ab$  into several (finitely many) subsegments, each subsegment lying completely in the interior or completely in the exterior. By cases 1 and 2, we know that the theorem holds for each subsegment. The theorem follows immediately for the whole segment as well.  $\square$

## 10 Source Unfoldings: Proof of Theorem 4

Recall that Fact 1 holds for Source Unfoldings as well. So, as in the proof of Theorem 3, if  $ab$  is completely inside the Source Unfolding then Theorem 4 follows immediately. The difficult case is when  $ab$  lies completely outside the Source Unfolding. The rest of this section will address this case. Once Theorem 4 is shown for this case, it will follow for the more general case (i.e.,  $ab$  being partly inside and partly outside) as well, exactly as in the proof of Theorem 3.

For the rest of this section, assume that  $ab$  lies completely outside the Source Unfolding. Both  $a$  and  $b$  must then be on the boundary of the unfolding. We will show that the following Lemma 6 holds in this case. Theorem 4 then follows from Lemma 6.



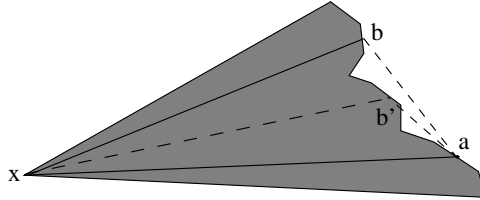
**Fig. 18.**  $source(ab) \geq SP(ab)$

**Lemma 6.** *If  $a, b$  are two points on the boundary of the Source Unfolding such that  $ab$  is completely outside the unfolding, then  $source(ab) \geq SP(ab)$ .*

**Proof:** Consider the triangle  $\triangle axb$  in the Source Unfolding and consider all the peels in the Source Unfolding which intersect this triangle. Without loss of generality, let the peels which intersect this triangle be  $p_1 \dots p_k$ , for some  $2 \leq k < m$  ( $m$  is the total number of peels in the Star Unfolding). The proof of Lemma 6 is by induction on  $k$ . Note that  $k \geq 2$ , otherwise (that is, when  $k = 1$ )

we have both  $a$  and  $b$  on the same peel; therefore  $ab$  lies entirely on a single peel  $p_1$  and hence,  $ab$  cannot lie outside the Source Unfolding at all.

**basis:**  $k = 2$ : Then we have  $a$  and  $b$  on the peels  $p_1$  and  $p_2$  respectively,  $v_1$  is the vertex common to the peels  $p_1$  and  $p_2$  and  $r_1$  is the ridge edge that is incident at  $v_1$ . Note that the three line segments  $v_1a$ ,  $r_1$  and  $v_1b$  are in the clockwise order around  $v_1$ . Let us cut the peel  $p_1$  of the Source Unfolding along  $xv_1$  and paste the peels  $p_1$  and  $p_2$  along the ridge edge  $r_1$ . This is the same as rotating the peel  $p_1$  with respect to the peel  $p_2$  as in the Source Unfolding to its position as in Star Unfolding with respect to the peel  $p_2$  as shown by the arrow in Fig.18. Thus when we move the peel  $p_1$  from its old position as in Source Unfolding to its new position as in Star Unfolding,  $v_1a$  turns clockwise around  $v_1$  by the same angle as is the curvature at  $v_1$ . Since the ridge edge incident at  $v_1$  bisects the two copies of  $x$  on the peels  $p_1$  and  $p_2$  in Star Unfolding, the peels  $p_1$  and  $p_2$  lie entirely on the opposite sides of  $r_1$ , and hence the clockwise order of  $v_1a$ ,  $r_1$  and  $v_1b$  is preserved after rotation. Since  $ab$  is outside the Source Unfolding, whereas  $v_1a$  and  $v_1b$  are inside, the angle  $\angle av_1b$  in the Source Unfolding is  $< \pi$ . Thus, as a consequence of the rotation, the angle  $\angle av_1b$  is smaller in the Star Unfolding when compared to the same angle in the Source Unfolding. Now compare the two triangles  $\Delta v_1ab$  in Source Unfolding and Star Unfolding. Since the two sides  $v_1a$  and  $v_1b$  are the same in both the triangles whereas the angle  $\angle av_1b$  is smaller in the Star Unfolding, the third side  $ab$  is smaller in the Star Unfolding. But the side  $ab$  in these triangles with respect to Source Unfolding and Star Unfolding are nothing but  $source(ab)$  and  $star(ab)$  respectively. Hence, we have  $source(ab) \geq star(ab) \geq SP(ab)$  in this case (the latter inequality due to Theorem 3).



**Fig. 19.** Existence of another copy  $b'$  of  $b$

**Induction:** Suppose we have proved the lemma for all  $i < k$  for some  $k > 2$ . Then we will show that the lemma holds for  $k$  as well, and it is as follows. Suppose we have the points  $a$  and  $b$  on the peels  $p_1$  and  $p_k$  respectively. Without loss of generality, we assume that the shortest path  $SP(xa)$  on the peel  $p_1$  is at least as large as the shortest path  $SP(xb)$  on the peel  $p_k$ . Since we assume that  $ab$  lies completely outside the Star Unfolding, according to Theorem 2 in Section 3, we have another shortest path  $SP(xb)$  on a peel  $p_i$ ,  $1 \leq i < k$ . Let us call this copy of  $b$  as  $b'$  (see Fig.19). We will show in the next two paragraphs

that  $source(ab) > source(ab')$  and  $source(ab') \geq SP(ab)$ . That would complete the proof of Lemma 6.

$source(ab) > source(ab')$ : Consider the two triangles  $\Delta axb$  and  $\Delta axb'$  (see Fig.19). We have the following.

- Since  $ab$  is outside the Star Unfolding, the angle  $\angle axb < \pi$ .
- Since the peel  $p_i$  is completely inside the triangle  $\Delta axb$ , the angle  $\angle axb' < \angle axb$ .
- The side  $ax$  is common to both the triangles.
- Since  $xb'$  lies on a single peel, it is a shortest path and hence,  $source(xb) = source(xb')$ .

Thus, we have  $source(ab) > source(ab')$ .

$source(ab') \geq SP(ab)$ :  $source(ab')$  can be split into maximal connected components that are completely on the Source Unfolding and that are completely outside the Source Unfolding. Consider any connected component, say  $cd$ . If  $cd$  lies completely on the Source Unfolding, then  $cd \geq SP(cd)$  according to Fact 1. Suppose  $cd$  is completely outside the Source Unfolding. Then  $cd$  must span fewer peels than  $k$  because  $i < k$ . By the induction hypothesis,  $cd \geq SP(cd)$ . Thus,  $source(ab') > SP(ab') = SP(ab)$ .  $\square$

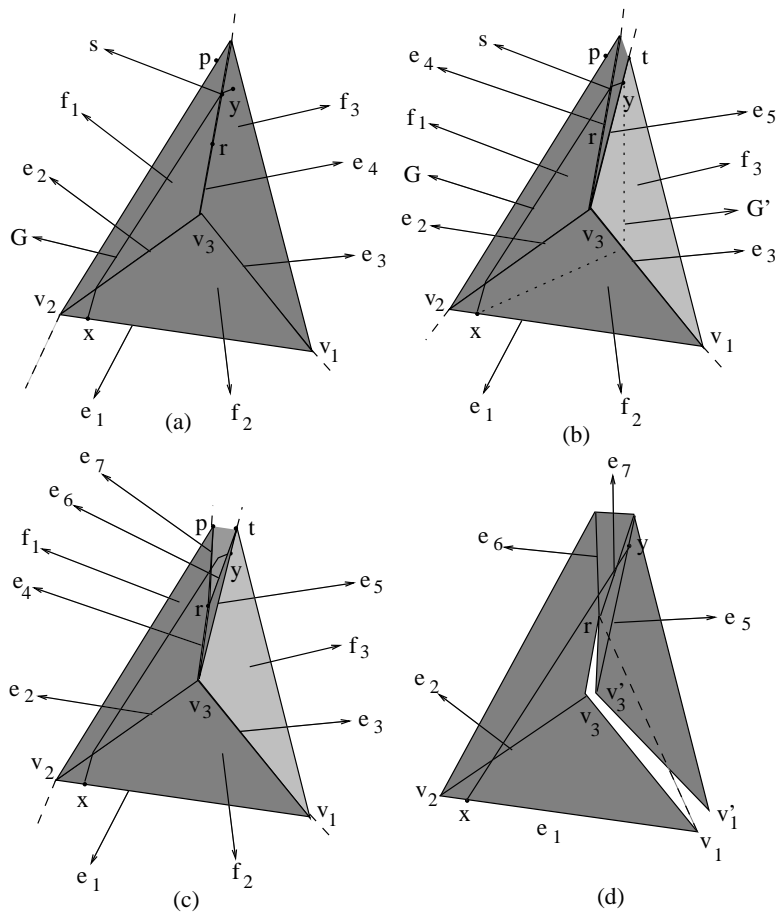
## 11 Planar Unfoldings can have Short-Cuts

Consider the Planar Unfolding with respect to a sequence of faces  $f_1 \dots f_k$  as defined in the introduction. As mentioned earlier, Planar Unfoldings could have short-cuts, i.e., there could be points  $(x, y)$  in the Planar Unfolding such that  $planar(xy) \leq SP(xy)$ . We provide an example for a short-cut in a Planar Unfolding below. It may be noted that an example for the overlap of Planar Unfolding is shown in [5]. Overlap of planar unfolding is a special case of a short-cut where the distance between the overlapping points  $x$  and  $y$  is equal to zero.

Let us construct a convex polyhedron whose unfolding has a short-cut  $(x, y)$  as follows. See Fig.20(a). The shaded portion is a ‘tetrahedral cap’ on a larger convex polyhedron. The tetrahedral portion has the edges  $e_1, e_2, e_3$  and  $e_4$  and three faces  $f_1, f_2$  and  $f_3$ . We assume that the curvature at the vertex  $v_3$  on the tetrahedron is very small. We assume that the rest of the polyhedron (excluding the tetrahedron) is very large so that we have a unique shortest path  $G = xy$  from a point  $x$  on the edge  $e_1$  to a point  $y$  on the face  $f_3$  as shown in the Fig.20(a).  $xy$  needs to be a shortest path because, we need the corresponding edge sequence to be a shortest path edge sequence. The shortest path  $G$  intersects the edges  $e_1, e_2$  and  $e_4$ . We also assume that  $x$  is arbitrarily close to  $v_2$  and  $y$  is arbitrarily close to the edge  $e_4$ . Let us now cut this polyhedron as follows.

- First, we cut the polyhedron along the plane through the three points  $v_3, v_1, y$ . We get a new edge  $e_5$  which passes through the points  $v_3$  and  $y$  as shown in the Fig.20(b). The geodesic  $G$  that was representing the shortest path from

$x$  to  $y$  before cutting remains unaltered in the new solid because, we are cutting the polyhedron along  $v_3y$  and  $v_3v_1$ . However,  $G$  need not be a shortest path in the new solid because, with a loss of a *slice* of the polyhedron due to the present cut, there can be a shorter path  $G'$  from  $x$  to  $y$  that traverses the new face created (as shown by a dotted curve in the Fig.20(b)). Since we can choose  $y$  as close to  $e_4$  as we want, the present cut can be as thin a slice as we desire. With arbitrarily thin slice being cut, one can assume that the geodesic  $G$  is a shortest path in the new solid as well.



**Fig. 20.** An example for a short-cut

- Let the shortest path  $G$  cross the edge  $e_4$  at  $s$ . Let  $r$  be any point on the edge  $e_4$  between  $s$  and  $v_3$ . Let  $t$  be any point on the edge  $e_5$  above  $y$  (that is,  $y$  lies on the portion  $v_3t$  of  $e_5$ ). Now let us cut this new solid again along

a plane that passes through  $r, t$  and  $p$ , where  $p$  lies on the face  $f_1$  above  $G$  (see the Fig.20(b)). Let  $e_6$  and  $e_7$  be the new edges that we get due to the second cut (see Fig.20(c)). Further, as a consequence of the second cut, the edge  $e_4$  is now restricted to its portion between  $v_3$  and  $r$ . We claim that the shortest path from  $x$  to  $y$  in the new solid traverses the new face that lies between the new edges  $e_6$  and  $e_7$  for the following reason. Contrary to the claim suppose that the shortest path from  $x$  to  $y$  in the new solid does not traverse the new face that lies between the edges  $e_6$  and  $e_7$ . Let  $G_1$  denote this shortest path from  $x$  to  $y$  on the new solid. Then  $G_1$  should lie on that portion of the new solid that was also part of the older solid (that is, before the present cut). Thus,  $G_1$  was a geodesic on the earlier solid as well. Since  $G$  was the shortest path from  $x$  to  $y$  on the earlier solid, we have  $G \leq G_1$ . Now let us superimpose the earlier solid and the present one. Consider the curve  $G$  that was on the older solid in relation to the new solid.  $G$  lies on the new solid everywhere except for a connected portion between the new edges  $e_6$  and  $e_7$  where it is off the new solid. We can shrink this portion like an elastic band so that this portion now lies on the new solid. Thus,  $G$  shrinks to a new curve  $G_2$  that completely lies on the new solid and hence, we have  $G_2 < G$ . Thus, we have  $G_2 < G \leq G_1$ . However, this leads us to the following contradiction. We have two curves  $G_1$  and  $G_2$  on the new solid between the points  $x$  and  $y$  where  $G_2 < G_1$ , but  $G_1$  is the shortest path. Thus, our claim that the shortest path from  $x$  to  $y$  in the new solid traverses the new face that lies between the new edges  $e_6$  and  $e_7$  holds.

Let us now cut this new solid shown in the Fig.20(c) along  $e_3$  and  $e_4$  and unfold it on a plane as in Fig.20(d). Because of the curvature at  $r$ , we have a wedge shaped gap between two copies of the edge  $e_4$ . Note that  $v_3$  and  $v_1$  are duplicated in the unfolding. Let  $v_3$  and  $v'_3$  be the two copies of  $v_3$  and  $v_1$  and  $v'_1$  be the two copies of  $v_1$ . We will show that  $v'_3v_1 < SP(v_3v_1)$ . Consider the three line segments  $rv_3, rv'_3$  and  $rv_1$  in the unfolding shown in the Fig.20(d). When the curvature of the polyhedron at the vertices  $v_3$  and  $r$  are sufficiently small, we have  $rv'_3$  between  $rv_3$  and  $rv_1$  (that is, we encounter  $rv'_3$  as we move anti-clockwise from  $rv_3$  to  $rv_1$ ). Consider the two triangles  $\Delta rv_3v_1$  and  $\Delta rv'_3v_1$ .  $rv'_3 = rv_3$  because they are the copies of the same edge.  $\angle v_3rv_1 > \angle v'_3rv_1$  because we have  $rv'_3$  between  $rv_3$  and  $rv_1$ . Hence, we have  $v_3v_1 > v'_3v_1$ . But,  $v_3v_1$  is a portion of an edge and hence, it is the unique shortest path between its end points. Thus,  $(v'_3, v_1)$  is shorter than  $SP(v_3v_1)$ , and it is a short-cut. In general, for chosen curvatures at  $v_3$  and  $r$ , we will have pairs of points  $(m, n)$  where  $m$  is in a small neighborhood of  $v'_3$ ,  $n$  is in a small neighborhood of  $v_1$  and  $(m, n)$  is a short-cut.